Tennessee State University

College of Engineering

Engineering Entrance Examination (EEE)

Review Guide

Version 2

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College of Engineering

Engineering Entrance Examination
(EEE)
Review Guide

The College of Engineering is committed to providing the highest quality curriculum and instruction to prepare students for a career in engineering and/or graduate school. The discipline requires a strong foundation in mathematics and physics, and a complementary background in chemistry. In this regard, all engineering students are required to take the Engineering Entrance Examination (EEE) "***before***" they enroll in any upper division engineering courses (3000 and 4000 level). In order to enroll in upper division engineering courses, the student must successfully passed the EEE with a minimum score of 75% on each part (calculus, physics, chemistry) of the examination. Listed below are eligibility requirements for taking the exam:

1. Minimum grade of "C" in each of the following courses: CHEM 1110, 1111; MATH 1910, 1920; PHYS 2110, 2111, 2120;
2. Minimum cumulative GPA of 2.5 and a minimum cumulative GPA of 2.5 for the group of courses listed above in Item 1 at the time of taking the Engineering Entrance Examination;
3. Completion and submission of the Engineering Entrance Examination eligibility form to the Dean's Office at least one (1) week prior to the examination.

The Engineering Entrance Examination is given at least five (5) times per year. The dates for the examination may be obtained from the Office of the Dean of the College of Engineering. Each student is allowed three (3) attempts to pass the Engineering Entrance Examination.

After the second unsuccessful attempt, the student is required to repeat at least one of the following courses: CHEM 1110; MATH 1910, 1920; PHYS 2110, 2120, before the examination can be taken a third and final time. Admission of transfer students also require taking and passing the Engineering Entrance Examination.

Failure to comply with any of the above requirements and guidelines may lead to dismissal of the program, or the additional enrollment of specific courses to satisfy the preparation for the examination.

The EEE Review Guide serves as a manual to prepare and pass the Engineering Entrance Examination. Through careful planning and an organized study schedule, you will substantially enhance your chances of passing the examination. ***The content of the review manual is solely for the purpose of preparing for the EEE,***

and is not an official document for publication or duplication for any other purpose.

Listed below is a schedule to help plan and study for the Engineering Entrance Examination. You should study for a minimum of eight (8) weeks before the exam.

DATE	SUBJECT	SECTION
WEEK 1		
WEEK 2		
WEEK 3		
WEEK 4		
WEEK 5		
WEEK 6		
WEEK 7		
WEEK 8		

On behalf of the College of Engineering, we look forward to your continued and successful matriculation in your engineering discipline of choice, and good luck on the EEE.

In Best Regards,

Dr. S. Keith Hargrove, Dean – College of Engineering

Disclaimer: The content of the Engineering Entrance Examination Review Guide is for the sole purpose of review for enrolled students in the College of Engineering at Tennessee State University. No content of this material should be duplicated, and is only for the purpose of review content for the examination.

MATHEMATICS

REVIEW

Part I: Algebra Review

A. Exponents

Basic rules:

1. $x^a \cdot x^b = x^{a+b}$
2. $\frac{x^a}{x^b} = x^{a-b}$
3. $(x^a)^b = x^{ab}$
4. $\frac{1}{x^a} = x^{-a}$ or $x^{-a} = \frac{1}{x^a}$
5. $\sqrt{x} = x^{\frac{1}{2}}$
6. $\sqrt[b]{x^a} = x^{\frac{a}{b}}$ or $x^{\frac{a}{b}} = \sqrt[b]{x^a}$

Below are several examples below

Example 1: $x^4 \cdot x^{-6} = x^{-2} = \frac{1}{x^2}$ We used rules 1 and 4.

Example 2: $\sqrt{x} \cdot \sqrt[3]{x} = x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} = x^{\frac{3+2}{6}} = x^{\frac{5}{6}}$ Rules 5, 6 and 1 were used, respectively

Example 3: $\frac{e}{e^3 \cdot e^{\frac{1}{2}}} = e \cdot e^{-3} \cdot e^{-\frac{1}{2}} = e^{\frac{2-6-1}{2}} = e^{-\frac{5}{2}} = \frac{1}{e^{\frac{5}{2}}} = \frac{1}{\sqrt{e^5}} = \frac{1}{\sqrt{e^4 e}} = \frac{1}{e^2\sqrt{e}}$

Rules 4, 1, 4, 6 were used and then the radical was simplified.

B. Factoring

When simplifying problems in mathematics the concept of factoring is needed often. Factoring is just a way of rewriting an expression that is added /subtracted as a multiplication problem or as factors.

There are some basics rules for factoring (which will be given later) but all methods of factoring cannot be expressed as a rule. Several examples are given below.

Common Monomial

This method should be tried first when asked to factor any expression.

Step 1: Find the largest common factor in each term.

Ex: $2x + 4y$ the largest common factor is 2.

Step 2: Rewrite each term as the product of the common factor time (multiplied by) some factor that makes up each of the original term. Here the common factor is 2.

$2x + 4y = \underline{2}(x) + \underline{2}(2y)$

Step 3: Rewrite the problem as a distributive problem with the common factor (written only once) outside the parenthesis and the sum/difference of the remaining terms inside the parenthesis.

$2(x + 2y)$

<u>More examples of common monomial factoring</u>:

Example 1: $2xy^2 - 8x = 2x(y^2) + 2x(-4) = 2x(y^2 - 4)$

Example 2: $3ab^2c^3 + 9a^3b^2c^4 + 6a^2b^2c^2$

$$\begin{aligned}&= 3ab^2c^3 + 9a^3b^2c^4 + 6a^2b^2c^2\\ &= 3ab^2c^2(c) + 3ab^2c^2(3a^2c^2) + 3ab^2c^2(2a)\\ &= 3ab^2c^2(c + 3a^2c^2 + 2a)\end{aligned}$$

Grouping

Use grouping when there are four or more terms in the polynomial.

Step 1: Group two or more terms together. With four terms, group two terms together.

$4x + 6xy \quad -9y - 6$

Step 2: Use the common monomial method to factor each group.

$2x(2 + 3y) \quad -3(3y + 2)$

Step 3: The factor in the parenthesis in each term are the same, so we can move the common factor out front and leave the remaining factor inside a parenthesis.

$$\begin{aligned}&2x(2 + 3y) - 3(3y + 2)\\ &= (2 + 3y)(2x - 3)\end{aligned}$$

More examples

Ex 1:

$3x^3 + 3x^2 + x + 1$

$=3x^2(x + 1)+1(x + 1)$

$=(x+1)(3x^2+1)$

Ex2: In the problem below there is a common factor in all terms so we should take out the common factor first.

Ex 2: Factor: $2txy + 2ctx - 3ty - 3ct$

$2txy + 2ctx - 3ty - 3ct$

$= t(2xy + 2cx - 3y - 3c)$ Now group and factor the expression inside the parenthessis.

$=t\left[2x(y+c) - 3(y+c)\right]$

$=t\left[(y+c)(2x-3)\right]$

Trinomials

A trinomial can be rewritten as a grouping problem and factored.

E: Factor $6x^2 + 5x - 6$

Step 1: Multiple the first and last terms together. $6x^2(-6)=-36x^2$

Step 2: Find two factors of $-36x^2$ whose sum equal the middle term.

$9x(-4x) = -36x$ and $+9x +(-4x) = 5x$, so our terms are $+9x$ and $-4x$.

Step 3: Replace the middle term in the original problem with these two terms then use the factoring by grouping method. In this case let: $+9x -4x = 5x$

$6x^2 + 9x - 4x - 6$

$=3x(2x + 3) - 2(2x +3)$

$=(2x + 3)(3x - 2)$

Ex: Factor: $x^2 - 5x - 6$

$$\begin{aligned} & x^2 - 5x - 6 \\ &= x^2 - 6x + 1x - 6 \\ &= x(x-6) + 1(x-6) \\ &= (x-6)(x+1) \end{aligned}$$

$-6x^2 = -6x(1x)$ and $-6x + 1x = -5x$

Basic rule for factoring polynomials

Examples will be given for 1 and 3 below, 2 below can be factored by the trinomial method above.

1. Difference of two squares: $x^2 - y^2 = (x-y)(x+y)$

   Try to rewrite the problem in the form of the rule above, if so, apply the rule.

   Ex 1: $x^2 - 36$

   $$\begin{aligned} x^2 - 36 &= x^2 - (6)^2 \\ &= (x+6)(x-6) \end{aligned}$$

   Ex 2: $49x^2 - 25z^2$

   $$\begin{aligned} 49x^2 - 25z^2 &= (7x)^2 - (5z)^2 \\ &= (7x-5z)(7x+5z) \end{aligned}$$

2. Trinomials that factor to a binomial square:

   a. $x^2 + 2xy + y^2 = (x+y)(x+y) = (x+y)^2$

   b. $x^2 - 2xy + y^2 = (x-y)(x-y) = (x-y)^2$

3. Sum and difference of Cubes:

   a. $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

   b. $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

   Ex 1: Factor: $x^3 - z^3$ This problem is in the form of rule b.

   $$x^3 - z^3 = (x-z)(x^2 + xz + z^2)$$

Ex 2: $8x^3 - 27$

$$8x^3 + 27 = (2x)^3 + 3^3$$
$$= (2x+3)\left[(2x)^2 - (2x)(3) + 3^2\right]$$
$$= (2x+3)(4x^2 - 6x + 9)$$

C. Solving Linear Equations

The below rules can be used to solve an equation for a variable not only in mathematics but in other courses. Some examples from Chemistry and Physics are included below in this section.

R1. Factor the denominator, if possible.

R2. Eliminate the denominator by finding LCD and multiplying it by each term. Cancel where possible.

R3. Remove all parentheses by multiplying, then simplify by combining *alike* terms.

R4. Get all *alike* terms on same side of the equal sign by using the inverse operation and simplify by combining *alike* terms.

R5. Divide by the number that is multiplied by the variable you are solving for.

R6. Cancel and/or reduce.

All steps may not apply. If a step doesn't apply, go to the next step. Now let's test the above steps to solve the linear equations below.

Example 1: Solve for x: $30 - 4 + 7x - 2 - 5x = 30$

$30 - 4 + 7x - 2 - 5x = 30$ Rules 1 – 3 do not apply. Using R4 we combine alike terms where possible.

$24 + 2x = 30$

$24 + 2x - 24 = 30 - 24$ To eliminate the –24 on the left side we should add the opposite +24 to both sides of the equation and simplify by combining alike terms

$2x = 6$ Rule 5 .

$x = 3$

Example 2: Solve for $\cos^2 x$:

$\sin^2 x + \cos^2 x = 1$ We are solving for $\cos^2 x$ therefore we need to eliminate the $\sin^2 x$ term by adding - $\sin^2 x$ to both side of the equation.

$$\sin^2 x + \cos^2 x - \sin^2 x = 1 - \sin^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

Example 3: Solve for x: $v^2 = v_0^2 + 2ax$

$v^2 = v_0^2 + 2ax$ Rule 4 : Need to eliminate the v_0^2 term.

$v^2 - v_0^2 = v_0^2 + 2ax - v_0^2$

$v^2 - v_0^2 = 2ax$ Use rule 5

$\dfrac{v^2 - v_0^2}{2a} = x$

Example 2: Solving linear equation with parentheses.

To solve an equation which contains one or more parentheses you must eliminate the parentheses by using the distributive law or multiplication of polynomials. We will examine these two cases.

Case 1: Solving an equation where the distributive law must be used.

Example 1

2(3x + 1) – 5(x + 2) = -9 Apply the distributive law for multiplication.

6x + 2 –5x – 10 = -9 Combine alike terms

x – 8 = -9 Eliminate the –8 by adding a + 8 to both sides

x – 8 + 8 = -9 +8

x = -1

Example 2:

2(x + 3) - (x + 1) = 5 Apply the distributive law.

2x + 6 – x – 1 = 5 Combine alike terms

x + 5 = 5

x + 5 – 5 = 5 – 5 Eliminating the + 5

x = 0

Example 3: Solve for v: $2x = (v_0 + v)t$

$2x = (v_0 + v)t$

$2x = v_0 t + vt$ R3

$2x - v_o t = vt$ R5

$\frac{2x - v_o t}{t} = v$

Case 2: Solving an equation where multiplication of polynomials should be used to eliminate the parentheses.

Example 1: $(x + 1)(x + 2) - 3x + 4 = 4x + (x - 3)(x + 3)$

$(x + 1)(x + 2) - 3x + 4 = 4x + (x - 3)(x + 3)$ Use the foil method to eliminate the parentheses.

$x^2 + 2x + 1x + 2 - 3x + 4 = 4x + x^2 - 9$ Combine alike terms

$x^2 + 6 = 4x + x^2 - 9$ Add - x^2 to both sides of the equation and simplify.

$x^2 + 6 - x^2 = 4x + x^2 - 9 - x^2$

$6 = 4x - 9$ Eliminate – 9 from the right side by adding + 9

$6 + 9 = 4x - 9 + 9$

$15 = 4x$ Use Rule 5

$\frac{15}{4} = x$

Equations with denominators

Example 1:

$\frac{(3x+4)}{3} - \frac{(x-2)}{2} = 3x - 7$ Rule 1: Does not apply

Rule 2: Find LCD. In this problem the LCD is 6.

We should multiply 6 by each term in the equation.

$6 \cdot \frac{(3x+4)}{3} - 6 \cdot \frac{(x-2)}{2} = 6 \cdot 3x - 7 \cdot 6$ Cancel where possible and simplify.

$2(3x+4) - 3(x-2) = 18x - 42$ Rule 3: Eliminate parentheses

$6x + 8 - 3x + 6 = 18x - 42$ Rule 4: Combine alike terms

$$3x+14=18x-42$$

$$3x+14-18x=18x-42-18x$$

$$-15x+14=-42$$

$$-15x+14-14=-42-14$$

$$-15x=-56$$ Rule 5:

$$x=\frac{56}{15}$$

Example 2: $\frac{1}{y+1}+\frac{7}{y^2-y-2}=\frac{3}{y-2}$

$\frac{1}{y+1}+\frac{7}{y^2-y-2}=\frac{3}{y-2}$ R1: Factor the denominator to find the LCD

$\frac{1}{y+1}+\frac{7}{(y+1)(y-2)}=\frac{3}{y-2}$ R2: The LCD is $(y+1)(y-2)$.

Before we go further we must restrict the solution(s) to values where the denominator(s) cannot equal zero. If the denominator is zero the problem will have no solution. With this problem we can easily see if y=-1 in the first fraction the denominator will equal zero. Also in the last fraction if y = 2 the problem will also have no solution. As we solve the equation we must place the following restrictions on y: $y \neq -1$ or 2.

$$(y+1)(y-2)\cdot\frac{1}{y+1}+(y+1)(y-2)\cdot\frac{7}{(y+1)(y-2)}=(y+1)(y-2)\cdot\frac{3}{y-2}$$

$1(y-2)+7=3(y+1)$ R3: Eliminate the parentheses.

$y-2+7=3y+3$ R3: Combine alike terms

$$
\begin{aligned}
y+5&=3y+3 \\
y+5-3y&=3y+3-3y && \text{R4} \\
-2y+5&=3 \\
-2y+5-5&=3-5 \\
-2y&=-2 && \text{R5} \\
y&=1
\end{aligned}
$$

$y \neq -1$ *or* 2, therefore the solution is $y = 1$.

D. Solving Systems of Equations

In this section we will solve a system of equation by two methods: a) the elimination and 2) the substitution methods. We will only solve systems of two equations with two unknown variable in this review. Solving a system of equations simply means to find the values of the variables that will make all the equations in a system true.

Elimination Method

In solving a system of equations by this method we try to eliminate one of the variables in order to solve for the others. A variable can be eliminated if the coefficients are the same number but signs are opposite. (Coefficients are normally referred to as the number in front of the variables, including the signs).

To solve a system:

1. If the two equations can be added so one variable will go to zero you have eliminated one variable. If not go to # 2 below.

    i) You have reduced the system to a linear equation where you can easily solve for one variable.

    ii) Solve for the other variable by plugging this answer in one of the original equations.

2. To solve for "x" – you eliminate the y-term (you can solve for y first in the same manner). To do so:
    i) Take the coefficient of the y-term in the first equation and multiply it (the coefficient) by the entire second equation.
    ii) Take the coefficient of the y-term in the second equation and multiply it by the entire first equation.
    iii) If both coefficient signs are the same, make one of these numbers negative before multiplying. If the signs are different make both numbers positive and multiply.
    iv) Add the two equations and solve for "x". The"y-term" should cancel out, that is it should become zero.
    v) You have reduced the system to a linear equation where you can easily solve for x.
    vi) To solve for *y* plug the value of *x* into one of the original equations.

Example 1: Solve the system for x and y given:

1. $3x + 2y = 0$
2. $5x + 3y = -1$

$3(3x + 2y = 0)$ -----> $9x + 6y = 0$ Rule 2

$-2(5x + 3y = -1)$-----> $-10x - 6y = 2$

$$9x + 6y = 0$$

$$\underline{-10x - 6y = 2}$$

$$-1x \quad = 2$$

$$\frac{-1x}{1} = \frac{2}{-1}$$

$$x = -2$$

Solving for y

$$3x + 2y = 0$$

$$3(-2)+2y = 0$$

$$-6 + 2y = 0$$

$$2y = 6$$

$$y = 3$$

Final Solution. $x = -2$, $y = 3$ or $(-2, 3)$

Example 2

$x + 3y = 0$

$20x - 15y = 75$

Solve for x

$15(x + 3y = 0)$ ----> $15x + 45y = 0$

$3(20x - 15y = 75)$ ------> $60x - 45y = 225$

$$15x + 45y = 0$$

$$\underline{60x - 45y = 225}$$

$$75x \quad = 225$$

$$\frac{75x}{75} = \frac{225}{75}$$

x = 3

Solve for y x + 3y = 0

3 + 3y = 0

3y = -3

y = -1

Final Solution: x = 3, y = -1 or (3, -1)

<u>Substitution method</u>

Solving systems by the substitution method

1. Select one of your equations and solve for either "x" or "y".
2. Substitute this expression (found in Step 1) for the appropriate variable in the other equation.
3. Solve for the value of the variable that remains.
4. To find the value of the other variable:
    i) Select one of the original equations
    ii) Substitute for the value found in Step 3
    iii) Solve for the remaining values.
5. Write your solution

Example 1: Solve the system below.

1. x – 3y = -7
2. x – 2y = 8

x – 3y = -7 Step 1: Take equation 1 and solve for x.

x = -7 + 3y

x – 2y = 8 Step 2: Substitute in equation 2

(-7 + 3y) – 2y = 8

-7 + 3y – 2y = 8 Step 3: The remaining variable is "y", so we must solve for "y"

3y – 2y = 8 + 7

y = 15

$x - 3y = -7$

Step 4: Replace y with 15 in equation 1, and solve for x.

$x - 3(15) = -7$

$x - 45 = -7$

$x = -7 + 45$

$x = 38$

Solution: x = 38 , y = 15 or (38, 15)

Example 2

1) $-5x + 2y = -6$
2) $10x + 7y = 34$

Solve for "y"

$-5x + 2y = -6$

$2y = 5x - 6$

$$\frac{2y}{2} = \frac{5x}{2} - \frac{6}{2}$$

$$y = \frac{5x}{2} - \frac{6}{2}$$

$$y = \frac{5x}{2} - 3$$

Now substitute in equation 2.

$10x + 7y = 34$

$$10x + 7\left(\frac{5x}{2} - 3\right) = 34$$

$$10x + \frac{35x}{2} - 21 = 34$$

Eliminate the fraction by multiplying by

$$10x(2) + \frac{35x}{2}(2) - 21(2) = 34(2)$$

$20x + 35x - 42 = 68$

Solve for x

$20x + 35x = 68 + 42$

55x = 110

x = 2

Now solve for y

10x + 7y = 34

10(2) + 7y = 34

20 + 7y = 34

7y = 34 – 20

7y = 14

y = 2 Solution: (2, 2)

E. The Equation of a Line

There are several approaches to find the equation of a line. The approach is determined by the given information.

A. Given two points: (x_1, y_1) and (x_2, y_2).

1. Find the slope of the line using the equation: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Use the point slope formula: $y - y_1 = m(x - x_1)$ and simplify.

Ex 1: Find the equation of the line given: $(-4,3)$ and $(3,-4)$

Step 1: $m = \dfrac{-4-3}{3--4} = \dfrac{-7}{7} = -1$

Step 2: $y - 3 = -1(x \ --4)$ Simplify

$$y = -1x - 4 + 3$$
$$y = -x - 1$$

B. Given slope m and a point (x_1, y_1).

Use the point slope formula: $y \ - y_1 = m(x - x_1)$

Ex: m=5 and point $(3,7)$ then

$$y - 7 = 5(x-3)$$
$$y = 5x - 15 + 7$$
$$y = 5x - 8$$

C. Given slope m and y-intercept b:

Use the slope intercept formula: $y = mx + b$

Ex: m =4 and b=9

$$y = mx + b$$
$$y = 4x + 9$$

Parallel Lines

Two lines are parallel if their slopes are the same

Ex 1: The lines $y = 4x + 9$ and $y = 4x - 2$ are parallel because their slopes m = 4.

Ex 2: Given $16x - 4y = 4$ and $y = 4x + 12$

If we solve equation 1 for y:

$$16x - 4y = 4$$
$$-4y = -16x + 4$$
$$y = 4x - 1 \qquad m = 4$$

Therefore the two equations are parallel.

Ex 3: Find the equation of a line *L* through point (-1, 3) parallel to line y = 7x – 3.

The slope of the given line is m =7. Since lines that are parallel have the same slopes, then the slope of line *L* is also 7.

m=7 and (-1, 3) We can use the point slope formula to find the parallel line *L*.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 7(x + 1)$$
$$y = 7x + 7 + 3$$
$$y = 7x + 10$$

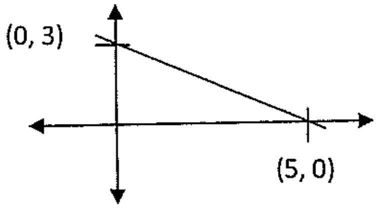
Perpendicular Lines

Two lines are perpendicular if their slopes are negative reciprocals or if the product of their slopes equal -1.

Ex 1: $y = -4x + 12$ and $y = \frac{1}{4}x + 1$

$m_1 = -4$ $\quad m_2 = \frac{1}{4}$ $\quad -4 \cdot \frac{1}{4} = -1$, therefore the two lines are perpendicular.

Ex 2: Find the equation of the line though the point (3, 3) and perpendicular to the given line *L*.



First we find the slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{5-0} = -\frac{3}{5}$

Perpendicular line slopes are negative recripocal of each other therefore the slope of the line we are seeking is

$$m = \frac{5}{3}$$

Using the point slope formula we get:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{5}{3}(x-3)$$

$$y - 3 = \frac{5}{3}x - 5$$

$$y = \frac{5}{3}x - 2 \text{ or}$$

$$5x - 3y = 6$$

F. Completing the Square

Completing the square is an algebraic concept that is needed not only in algebra but in calculus, differential equations and other areas. In algebra completing the square can be used to solve a quadratic equation. In precalculus this process is used to write the general equation of conic equation in standard form. In calculus and differential equations completing the square is used in integration, Laplace transforms and many areas in mathematics.

Use completing the square to solve for x in a quadratic equation.

Step 1: Write the quadratic equation in the form: $ax^2 + bx = c$

Step 2: If *a* does not equal 1, divide each term in the equation by *a*.

Step 3: To the side -Multiply the new coefficient of the x –term by ½, square the answer and then add this answer to both sides of the equation and simplify.

Step 4: The left-hand side of the equation should factor as a binominal squared.

Step 5: Take the square root of both sides of the equation and solve for x.

Example 1: $x^2 - 4x = 6$

$$\begin{aligned} x^2 - 4x &= 6 \\ x^2 - 4x + 4 &= 6 + 4 \\ x^2 - 4x + 4 &= 10 \\ (x-2)^2 &= 10 \end{aligned} \qquad \frac{1}{2}(-4) = (-2)^2 = 4$$

$$\begin{aligned} \sqrt{(x-2)^2} &= \sqrt{10} \\ x - 2 &= \pm\sqrt{10} \\ \mathrm{x} &= 2 \pm \sqrt{10} \end{aligned}$$

Write the general equation of a parabola in standard form and find the vertex:

Standard form: $y = a(x-h)^2 + k$ with vertex: (h, k).

Given: $f(x) = ax^2 + bx + c$ General form

Step 1: If *a* is not 1, divide each term by *a*.

$$f(x) = \frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a}$$

Step 2: Multiply the new coefficient of the x –term by ½ and square the answer. Move the constant term over because we will add and subtract the results in the next step.

$$f(x) = x^2 + \frac{b}{a}x \qquad + \frac{c}{a} \qquad\qquad \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$

Step 3: Add the results from step 2 to the first two term and subtract from the constant term. This does not change the value of the expression.

$$f(x) = x^2 + \frac{b}{a}x + \underline{\frac{b^2}{4a^2}} \; + \frac{c}{a} - \underline{\frac{b^2}{4a^2}}$$

Step 4: The first three terms should factor as a binomial squared. Combine the last two terms.

$$f(x) = (x + \frac{b}{2a})^2 \;\; + \;\; (\frac{b^2 - 4ac}{4a^2})$$

Example 1: $f(x) = x^2 - 6x + 6$

Step 1: In this case $a = 1$, so we can skip this step

$$f(x) = x^2 - 6x + 6$$

Step 2: Multiply the new coefficient of the x –term by ½ and square the answer. Move the constant term over because we will add and subtract the results in the next step.

$$f(x) = x^2 - 6x \qquad + 6 \qquad\qquad \left(\frac{1}{2}\cdot - 6\right)^2 = (-3)^2 = 9$$

Step 3: Add this term to the first two term and subtract from the constant term. This does not change the value of the expression.

$$f(x) = x^2 - 6x \;\; + \underline{9} \;\; + 6 - \underline{9}$$

$$= (x^2 - 6x \;\; + 9) \;\; + (6 - 9)$$

Step 4: The first three terms should factor as a binomial square. Combine the last two terms.

$f(x) = (x-3)^2 - 3$ Vertex: (3, -3)

Write the general equation of a circle in standard form and find the center and radius

$y = (x-h)^2 + (x-k)^2 = r^2$ Center (h, k) radius =r Standard form

$x^2 + 2y^2 + 2x - 20y + 2 = 0$ General form

Step 1: Group the x-term and y-terms together, take the constant to the opposite side of the equation.

$$x^2 + 2x \quad + y^2 - 10y \quad = -2$$

Step 2: Use the rules for the algebra method above to complete the square on both the x and y terms.

$$x^2 + 2x + \underline{1} + y^2 - 10y + \underline{25} = -2 + \underline{1} + \underline{25}$$

$(x+1)^2 + (y-5)^2 = 24$ Standard form. Center $(-1, 5)$, radius $= \sqrt{24}$

G. Evaluating Determinants

Determinants are used in varies areas of math. It is a tool that can simplify more complex problem. Here we will only evaluate **2 X 2** and **3 X 3** determinants, higher orders can easily be computed on a calculator or computer.

An example of a **2 X 2 and 3 X 3** determinants are given below. Be sure not to get it confused with a matrix. We evaluate determinants, with matrices we perform operations (add, subtract, etc.).

2 X 2 = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ 3 X 3 = $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

Rule for evaluating a 2 X 2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

This simply states: The product of the terms in the a and d positions minus the product of the terms in the b and c positions.

Example 1:

$$\begin{vmatrix} 5 & 2 \\ -2 & -1 \end{vmatrix} = (5)(-1) - (2)(-2) = -5 + 4 = -1$$

A **3 X 3** determinant can be evaluate in several ways. We will show two ways in this review.

Method 1: Cofactor Method

In this method you select any row or column and find the **minor** for each component in the selected row /column. The minor consist of all components not in the same row or column with the selected component row / column. If we are evaluating a 3 X 3 determinant the minor will consist of a 2 X 2 determinant. The cofactors can be found as given below.

Determinant	Minor	Cofactor
$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$	$M_{11} = \begin{vmatrix} b_{22} & b_{23} \\ c_{32} & c_{33} \end{vmatrix}$	$A_{11} = (-1)^{1+1} M_{11}$
$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$	$M_{12} = \begin{vmatrix} b_{21} & b_{23} \\ c_{31} & c_{33} \end{vmatrix}$	$A_{12} = (-1)^{1+2} M_{12}$
$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$	$M_{13} = \begin{vmatrix} b_{21} & b_{22} \\ c_{31} & c_{32} \end{vmatrix}$	$A_{13} = (-1)^{1+3} M_{13}$

The value of the determinant $|A|$ is the sum of the cofactors:

$$|A| = (-1)^{1+1} M_{11} + (-1)^{1+2} M_{12} + (-1)^{1+3} M_{13}$$

Example 1:

$$\begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 5 & -3 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -3 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 \\ 5 & -3 \end{vmatrix}$$

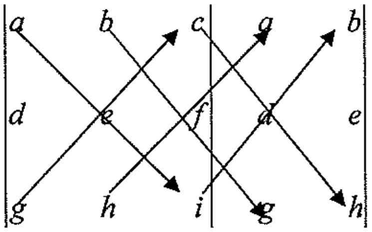
$$\begin{aligned} &= 1(-2+3) - 3(0-5) - 1(0+10) \\ &= 1 + 15 - 10 \\ &= 6 \end{aligned}$$

Method 2:

Step 1: Add the first two columns to the end of the determinant as given below.

$$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right| =$$

Step 2: Find the product of the terms along the diagonals (as indicated by the lines). The product of the terms along the arrows down should be added and the product of the terms arrows going up should be subtracted.



$$= aei + bfg + cdh - gec - hfa - idb$$

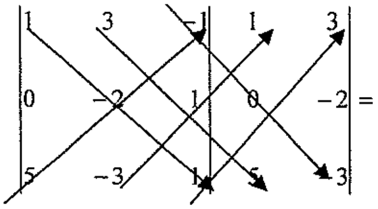
Example: Evaluate the determinant below:

$$\begin{vmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 5 & -3 & 1 \end{vmatrix}$$

Step 1:

$$\left|\begin{array}{ccc|cc} 1 & 3 & -1 & 1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 5 & -3 & 1 & 5 & -3 \end{array}\right| =$$

Step 2:



We take the sum of the product of the values on the arrows from top to bottom (going down) and subtract the produce of the arrows from bottom to top (going up).

$$= (1)(-2)(1)+(3)(1)(5)+(-1)(0)(-3)-(5)(-2)(-1)-(-3)(1)(1)-(1)(0)(3)$$
$$= -2+15+0-10+3-0$$
$$= 6$$

H. Natural Log functions

Basic rules for simplifying natural log functions

1. $\ln u + \ln w = \ln(u \cdot w)$
2. $\ln u - \ln w = \ln\left(\frac{u}{w}\right)$
3. $u \cdot \ln w = \ln w^u$

Ex 1:

$$y = \ln(x-2) + \ln(x+5)$$
$$= \ln[(x-2)(x+5)]$$

Ex 2:

$$y = \ln(x-2) - \ln(x+5)$$
$$= \ln\left(\frac{x-2}{x+5}\right)$$

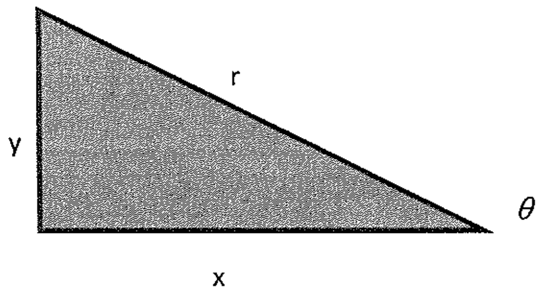
Ex 3:

$$y = 3\ln(x-2)$$
$$= \ln(x-2)^3$$

Part II: Trigonometric Functions

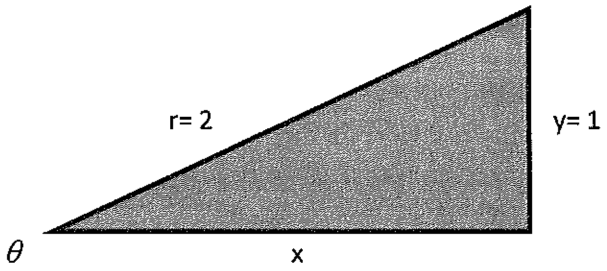
Trigonometric functions can be defined as circular functions or by using the right triangle. We will define these functions in turns of the right triangle. Given the right triangle:

A. The Right Triangle



1. $Sin\theta = \frac{opp}{hyp} = \frac{y}{r}$
2. $Cos\theta = \frac{adj}{hyp} = \frac{x}{r}$
3. $Tan\theta = \frac{opp}{adj} = \frac{y}{x}$
4. $Csc\theta = \frac{hyp}{opp} = \frac{r}{y}$
5. $Sec\theta = \frac{hyp}{adj} = \frac{r}{x}$
6. $Cot\theta = \frac{adj}{opp} = \frac{x}{y}$

Note: The two functions across from each other are reciprocal functions.



Find the six trig functions for the right triangle above.

First we should find the length of the missing side using the Pythagoras theorem.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ 2^2 &= x^2 + 1^2 \\ 4 - 1 &= x^2 \\ \sqrt{3} &= x \end{aligned}$$

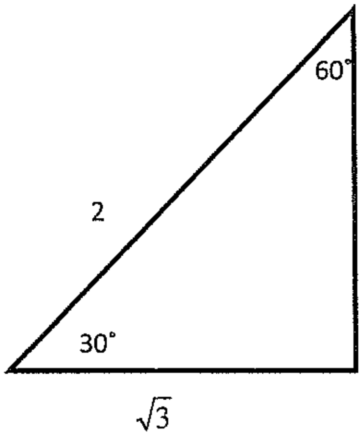
Using the rules above we can find all six trig functional values:

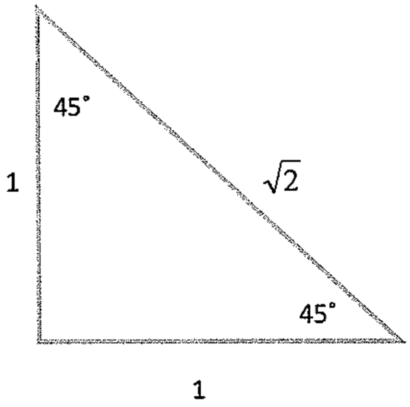
1. $Sin\theta = \frac{opp}{hyp} = \frac{1}{2}$
2. $Cos\theta = \frac{adj}{hyp} = \frac{\sqrt{3}}{2}$
3. $Tan\theta = \frac{opp}{adj} = \frac{1}{\sqrt{3}}$
4. $Csc\theta = \frac{hyp}{opp} = \frac{2}{1}$
5. $Sec\theta = \frac{hyp}{adj} = \frac{2}{\sqrt{3}}$
6. $Cot\theta = \frac{adj}{opp} = \frac{\sqrt{3}}{1}$

Students need to know how to find the six trig functions for the following angles:

$30^\circ, 45^\circ, 60^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$

You can find the values of the six trig functions for the angles by knowing the two basic triangles below and the angles that





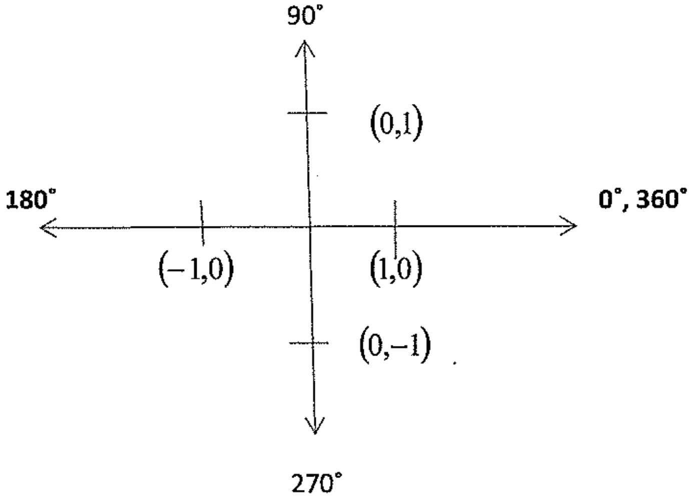
For 30°: opp = 1, adj = $\sqrt{3}$, hyp = 2

For 45°: opp = 1, adj = 1, hyp = $\sqrt{2}$

For 60° : opp = $\sqrt{3}$, adj = 1, hyp = 2

Example: For the 45° angle the values of the six trig functions are as following:

1. $Sin\ 45^\circ = \frac{opp}{hyp} = \frac{1}{\sqrt{2}}$
2. $Cos 45^\circ = \frac{adj}{hyp} = \frac{1}{\sqrt{2}}$
3. $Tan\ 45^\circ = \frac{opp}{adj} = \frac{1}{1} = 1$
4. $Csc\ 45^\circ = \frac{hyp}{opp} = \sqrt{2}$
5. $Sec\ 45^\circ = \frac{hyp}{adj} = \sqrt{2}$
6. $Cot\ 45^\circ = \frac{adj}{opp} = 1$



For the angles on the axes the x- coordinate represents the value cosine and the y coordinate represent the sine. $(\cos\theta, \sin\theta)$.

For: 0° and 360 ° the coordinate is $(1,0)$

$\sin 2\pi = 0, \quad \cos 2\pi = 1, \quad \tan 2\pi = \dfrac{\sin\theta}{\cos\theta} = \dfrac{0}{1} = 0$ See theidentitiesrulesbelow.

$\csc 2\pi = \dfrac{1}{0} = \infty, \quad \sec 2\pi = 1, \quad \tan 2\pi = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{0} = undefined$

For: 90°=$\frac{\pi}{2}$ coordinate is $(0,1)$

$\sin\frac{\pi}{2} = 1, \quad \cos\frac{\pi}{2} = 0, \quad \tan\frac{\pi}{2} = \frac{\sin\theta}{\cos\theta} = \frac{1}{0} = undefined$

$\csc\frac{\pi}{2} = 1, \quad \sec\frac{\pi}{2} = undefined \quad \tan\frac{\pi}{2} = \frac{\cos\theta}{\sin\theta} = 0$

For all the other angles on the axes the trig function values can be found in this manner.

B. Fundamental Trigonometric Identities

Reciprocal identities

1. $Sin\theta = \dfrac{1}{\csc\theta}$
2. $Cos\theta = \dfrac{1}{\sec\theta}$
3. $Tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$
4. $Csc\theta = \dfrac{1}{\sin\theta}$
5. $Sec\theta = \dfrac{1}{\cos\theta}$
6. $Cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

Double angle identities: 1. $\sin 2\theta = 2\sin\theta\cos\theta$ 2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Half angle identities: 1. $\sin^2 u = \dfrac{1-\cos 2u}{2}$ 2. $\cos^2 u = \dfrac{1+\cos 2u}{2}$

In calculus students sometimes will have to change the form of some trig functions in order to perform certain operations. This section will change the form of several trig expressions to other forms.

Example 1: Show that the statement is true: $\sin u + \cos u \cot u = \csc u$

$$\sin u + \cos u \cot u = \csc u$$

$$\sin u + \cos u \cdot \frac{\cos u}{\sin u} =$$

Rewrite cot u in terms of sin u and cos u and simplify

$$\sin u + \frac{\cos^2 u}{\sin u} =$$

Find the LCD and add

$$\frac{\sin^2 u + \cos^2 u}{\sin u} =$$

Use the identity $\sin^2 u + \cos^2 u = 1$

$$\frac{1}{\sin u} =$$

Use the trig identity $\dfrac{1}{\sin u} = \csc u$

$$\csc \text{u} = \csc u$$

Example 2: $(\cos^2 u)(\sec^2 u) =$

$$(\cos^2 u)\left(\frac{1}{\cos^2 u}\right) =$$

$$1 = 1$$

Example 3:

$$\cos^2 x(\sec^2 x - 1) = \sin^2 x$$

$$\cos^2 x \tan^2 x =$$

$$\cos^2 x \cdot \frac{\sin^2 x}{\cos^2 x} =$$

$$\sin^2 x = \sin^2 x$$

Example 4:

$$\frac{\csc x}{\sec x} = \cot x$$

$$\frac{\frac{1}{\sin x}}{\frac{1}{\cos x}} =$$

$$\frac{1}{\sin x} \cdot \frac{\cos x}{1} =$$

$$\frac{\cos x}{\sin x} =$$

$$\cot x = \cot x$$

Part III: Calculus

A. Limits

Algebraic Method

Definition 1: The limit of a constant is that constant:

Ex: $\lim_{x \to c}(a) = a$ whenever 'a' is a constant

Ex. 1 $\lim_{x \to 2}(-5) = -5$ Ex. 2 $\lim_{x \to 0}(35) = 35$

To find the limit of a function f(x) as x approach some **constant "c"** ($x \to c$), try the methods given below, in the order that they are given.

Method 1:

A. Replace the value of x with the value of c throughout the equation.
B. Simplify the expression
C. If you get a numerical answer, then that is the limit.
D. If you get one of the following forms you must try a different method

You cannot use this method if you get: $\frac{\infty}{\infty}$ *or* $\frac{0}{0}$, which are indeterminate forms.

Example 1: Find the limit given: $\lim_{x \to 2}(2x - 5) =$

$$\lim_{x \to 2}(2x-5) = \lim_{x \to 2}\left[2(2)-5\right]$$

$$= \lim_{x \to 2}(4-5)$$

$$= \lim_{x \to 2}(-1)$$

$$= -1$$

Example 2: Find the limit given: $\lim_{x \to 3} \sqrt{7+3x}$

$$\lim_{x \to 3} \sqrt{7+3x} = \lim_{x \to 3} \sqrt{7+3(3)}$$

$$= \lim_{x \to 3} \sqrt{7+9}$$

$$= \lim_{x \to 3} \sqrt{16} = \lim_{x \to 3} 4 = 4$$

Example 3: Find the limit given: $\lim_{x\to 3} \frac{x-4}{2x^2+5x-3}$

$$\lim_{x\to 3} \frac{x-4}{2x^2+5x-3} = \lim_{x\to 3} \frac{(3)-4}{2(3)^2+5(3)-3}$$

$$= \lim_{x\to 3} \frac{-1}{18+15-3}$$

$$= \lim_{x\to 3} \frac{-1}{30}$$

$$= -\frac{1}{30}$$

Ex 4: $\lim_{x\to -2} \frac{x^2+2x-3}{x^2+5x+6} = \lim_{x\to -2} \frac{-4+4-3}{(-2)^2+5(-2)+6}$

$$= \lim_{x\to -2} \frac{-3}{0}$$

$= \text{DNE}$ Does not exists

Method 2: Redefine (rewrite in a different form) the function method. Try this method if you get one of the following forms: $\frac{\infty}{\infty}$ *or* $\frac{0}{0}$, which are indeterminate forms.

Ex 1: $\lim_{x\to 4} \frac{x-4}{x^2-16} = \lim_{x\to 4} \frac{(4)-4}{(4)^2-16}$

$$= \lim_{x\to 4} \frac{0}{0}$$

Since this is an indeterminate form, you must rewrite the function. In this case you can factor and reduce the expression.

$$\lim_{x\to 4} \frac{x-4}{x^2-16} = \lim_{x\to 4} \frac{(x-4)}{(x-4)(x+4)}$$

$= \lim_{x\to 4} \frac{1\cdot(x-4)}{(x-4)(x+4)}$ Keep in mind the numerator can be written as given here

$= \lim_{x\to 4} \frac{1}{x+4}$ Find the limit as in method 1.

$$= \lim_{x\to 4} \frac{1}{(4)+4}$$

$$= \lim_{x\to 4} \frac{1}{8}$$

$$= \frac{1}{8}$$

Ex 2: $\lim_{x\to 3^+} \frac{x^2-3x}{x^2-5x+6} = \lim_{x\to 3^+} \frac{3^2-3(3)}{3^2-5(3)+6}$

$$= \lim_{x\to 3^+} \frac{0}{0}$$

An indeterminate form, so try to factor and reduce the expression.

$$\lim_{x\to 3^+} \frac{x^2-3x}{x^2-5x+6} = \lim_{x\to 3^+} \frac{x(x-3)}{(x-2)(x-3)}$$

$$= \lim_{x\to 3^+} \frac{x}{(x-2)}$$

$$= \lim_{x\to 3^+} \frac{3}{(3-2)}$$

$$=3$$

Ex 3: $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(4)-4}{\sqrt{4}-2}$

$$= \lim_{x\to 4} \frac{0}{0}$$

Since this is an indeterminate form, you must again rewrite the function. In this case some student cannot factor this problem correctly. An alternate approach is to rationalize the denominator by multiplying by the conjugate, simplify and use method 1 to find the limit.

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

Use the foil method to multiply the denominator only.

$$= \lim_{x\to 4} \frac{(x-4)\left(\sqrt{x}+2\right)}{x+2\sqrt{x}-2\sqrt{x}-4}$$

$$= \lim_{x \to 4} \frac{(x-4)\left(\sqrt{x}+2\right)}{(x-4)}$$ Simplify and find the limit as in Method 1 above.

$$= \lim_{x \to 4} \left(\sqrt{x}+2\right)$$

$$= \lim_{x \to 4} \left(\sqrt{4}+2\right)$$

$$= 4$$

Ex 4: $\lim_{x \to 0} \frac{(h+3)^2-9}{h} \quad = \quad \lim_{x \to 0} \frac{(0+3)^2-9}{0} = \frac{0}{0}$

This expression yields an indeterminate form also. Here it would be best to expand the binominal to simplify the numerator.

$$= \lim_{h \to 0} \frac{h^2+6h+9-9}{h}$$ Simplifying we get

$$= \lim_{h \to 0} \frac{h^2+6h}{h}$$ Factor

$$= \lim_{h \to 0} \frac{h(h+6)}{h}$$ Reduce

$$= \lim_{h \to 0} (h+6)$$

$$= 6$$

Let's find the limits using trig functions

Ex 5: $\lim_{x \to 0} \frac{1-\cos x}{\sin x} = \lim_{x \to 0} \frac{1-\cos(0)}{\sin(0)}$

$$= \lim_{x \to 0} \frac{1-1}{0} = \lim_{x \to 0} \frac{0}{0}$$ indeterminate form

Rationalize the numerator

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x}\cdot\frac{(1+\cos x)}{(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x(1+\cos x)}$$

Use a trig identity ($\sin^2 x=1-\cos^2 x$)

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin x(1+\cos x)}$$

Reduce

$$=\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin(0)}{(1+\cos(0))}$$

$$=\lim_{x\to 0}\frac{0}{(1+1)}$$

$$=0$$

To save some time we will use the following definition:

Definition: $\lim_{x\to 0}\frac{\sin\theta}{\theta}=1$, whenever the angle for sine is the same as the angle in the denominator this statement is true. The reciprocal is also true: $\lim_{x\to 0}\frac{\theta}{\sin\theta}=1$.

Find the limit of: $\lim_{x\to 0}\frac{\sin 3\theta}{\theta}=$

The **angle for sine is different from the angle in the denominator**. At this point we cannot use the rule. We must try to make our denominator 3θ. At this point we have an indeterminate form.

$$=\lim_{x\to 0}\frac{\sin 3(0)}{(0)}$$

$$=\lim_{x\to 0}\frac{0}{(0)}$$

Let's use some algebra to get 3θ in the denominator. If we multiply our expression by 1 we will not change the value of the problem. But we will let

$$1 = \frac{3\theta}{3\theta}$$

$$\lim_{x\to 0} \frac{\sin 3\theta}{\theta} = \lim_{x\to 0} \frac{\sin 3\theta}{\theta}\cdot 1$$

$$= \lim_{x\to 0} \frac{\sin 3\theta}{\theta}\cdot\frac{3\theta}{3\theta}$$ Simplify

$$= \lim_{x\to 0} \frac{\sin 3\theta}{}\cdot\frac{3}{3\theta}$$

$$= \lim_{x\to 0} \frac{3\cdot\sin 3\theta}{3\theta}$$ Using definition 1 above we get

$$= \lim_{x\to 0}\ 3\cdot 1$$

$$= 3$$

Ex 6: $\lim\limits_{x\to 0} \dfrac{2x}{\tan x} \quad = \dfrac{0}{0}$

If we rewrite $\tan x = \dfrac{\sin x}{\cos x}$ we get

$$\lim_{x\to 0} \frac{2x}{\tan x} \quad = \lim_{x\to 0} \frac{2x}{\frac{\sin x}{\cos x}}$$ Invert and multiply the denominator

$$= \lim_{x\to 0} \frac{2x\cdot\cos x}{\sin x}$$

$$= \lim_{x\to 0} \frac{2\cos x}{1}\cdot\lim_{x\to 0}\frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{2\cos x}{1}\cdot 1$$

$$= \lim_{x\to 0}\ 2\cos x$$

$$= \lim_{x\to 0}\ 2\cdot\cos(0)$$

= 2

Example 7: $\lim_{x\to 0}\frac{\sin x}{1-\cos x} = \lim_{x\to 0}\frac{\sin(0)}{1-\cos(0)}$

$$= \lim_{x\to 0}\frac{0}{0}$$

We can solve this problem by multiplying by the conjugate to solve this problem

$$\lim_{x\to 0}\frac{\sin x}{1-\cos x} = \lim_{x\to 0}\frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x(1+\cos x)}{1-\cos^2 x}$$

$$= \lim_{x\to 0}\frac{\sin x(1+\cos x)}{\sin^2 x}$$ Simplifying we get

$$= \lim_{x\to 0}\frac{(1+\cos x)}{\sin\ x}$$

$$= \lim_{x\to 0}\frac{(1+1)}{0} = \frac{2}{0} = \textbf{DNE}$$

B. Limits as $x\to\infty$ and rational functions: $\lim_{x\to\infty}\frac{f(x)}{g(x)}$

The Algebraic Approach

Before we use this approach we will use the following expression as a definition for time sake.

Definition: $\lim_{x\to\infty}\frac{c}{x^p}\to 0$ where c is any constant (number) and p is a positive number.

Example: Find the limit

$$\lim_{x\to\infty}\frac{5}{x^3}$$

$$\lim_{x\to\infty}\frac{5}{\infty^3} = \frac{5}{\infty} = 0$$

To find the limit of a given a rational function where $x\to\infty$ we can use the following steps to determine the limit.

1. Find the highest degree term in the given rational function.
2. Divide each term in the rational polynomial by the highest degree term

3. Reduce (simplify) each individual terms.

4. Replace all remaining x-values with ∞.

5. Use the definition above

Ex 1: $\lim\limits_{x\to\infty} \dfrac{x^3}{x^9+2}$

Step 1: The highest degree term is: x^9

Step 2: Divide each term in the polynomial by this term and simplify.

$$\lim_{x\to\infty} \frac{\dfrac{x^3}{x^9}}{\dfrac{x^9}{x^9}+\dfrac{2}{x^9}}$$

Step 3 $$\lim_{x\to\infty} \frac{\dfrac{1}{x^6}}{1+\dfrac{2}{x^9}}$$

Step 4 $$\lim_{x\to\infty} \frac{\dfrac{1}{\infty^6}}{1+\dfrac{2}{\infty^9}} = \frac{0}{1+0} = 0$$

Ex 2: $$\lim_{x\to\infty} \frac{x^3-x^2+2x^{23}-1}{x^9+2x^4+1} = \lim_{x\to\infty} \frac{x^3-x^2+2x^{23}-1}{x^9+2x^4+1}$$

$$= \lim_{x\to\infty} \frac{\dfrac{x^3}{x^{23}}-\dfrac{x^2}{x^{23}}+\dfrac{2x^{23}}{x^{23}}-\dfrac{1}{x^{23}}}{\dfrac{x^9}{x^{23}}+\dfrac{2x^4}{x^{23}}+\dfrac{1}{x^{23}}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{x^{20}}-\dfrac{1}{x^{21}}+\dfrac{2}{1}-\dfrac{1}{x^{23}}}{\dfrac{1}{x^{14}}+\dfrac{2}{x^{19}}+\dfrac{1}{x^{23}}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\infty^{20}} - \frac{1}{\infty^{21}} + \frac{2}{1} - \frac{1}{\infty^{23}}}{\frac{1}{\infty^{14}} + \frac{2}{\infty^{19}} + \frac{1}{\infty^{23}}}$$

$$= \lim_{x\to\infty} \frac{0-0+2-0}{0+0+0} \quad = \frac{2}{0} = \infty$$

Ex 3: $\lim_{x\to\infty} \frac{4x^3 - 1}{2x - x^3 + 11}$ $\quad = \lim_{x\to\infty} \frac{\frac{4x^3}{x^3} - \frac{1}{x^3}}{\frac{2x}{x^3} - \frac{x^3}{x^3} + \frac{11}{x^3}}$

$$= \lim_{x\to\infty} \frac{4 - \frac{1}{x^3}}{\frac{2}{x^2} - 1 + \frac{11}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4-0}{0-1+0} \quad = -4$$

This method can be quite long, but there is a short cut for this method.

Short Cut

To find the limit of a **<u>rational function as</u>** $x \to \infty$**, you can used the short cut below.**

1. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ if the highest degree term is in the denominator. (See example 1 above).

2. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ if the highest degree term is in the numerator. (See example 2 above).

3. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{c_n}{c_d}$ if the highest degree term is in both the denominator and numerator the limit will be the coefficients of these terms. (See example 3 above).

Let's use the short cut on the problems below:

Ex 1: $\lim_{x\to\infty} \frac{x-2}{x^3-8} = 0$

Ex 2: $\lim_{x\to\infty} \frac{x^2 - 6x^4 - 3}{-x^4 + 2x} = \frac{-6}{-1} = 6$

Ex 3: $\lim_{x\to\infty} \frac{x^2+4x-5x^7}{x^2+x-2} = \infty$

C. L'Hopital's Rule

If a limit results in an indeterminate form another approach can be used to determine if the limit exists.

L'Hopital's Rule: The rule states that the limit of a quotient of two function is the same as the limit of the derivatives of the two functions. L'Hopital's should only be used when with the two indeterminate forms are obtained:

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

L'Hopital's Rule: $\lim_{x\to c} \frac{u}{v} = \lim_{x\to c} \frac{u'}{v'}$

Ex 1: $\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \frac{3^3-2(3)-3}{3-3} = \frac{0}{0}$ We can apply L'Hopital's Rule here.

We find the derivative of the numerator and the derivative of the denominator. (<u>Do not use the quotient rule)</u>.

$$\lim_{x\to 3} \frac{\frac{d}{dx}(x^2-2x-3)}{\frac{d}{dx}(x-3)} = \lim_{x\to 3} \frac{2x-2}{1} = \frac{2(3)-2}{1} = \frac{4}{1} = 4$$

Ex 2: $\lim_{x\to\infty} \frac{\ln x}{e^{2x}} = \frac{\ln\infty}{e^{\infty}} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\ln x}{e^{2x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2e^{2x}} = \lim_{x\to\infty} \frac{1}{2xe^{2x}} = \frac{1}{\infty} = 0$$

Other indeterminate forms:

1. $\lim\limits_{x\to 0^+} u\cdot v = 0\cdot\infty$, if you get this indeterminate form rewrite the limit as $\lim\limits_{x\to 0^+} \frac{v}{\frac{1}{u}}$

Ex : $\lim\limits_{x\to 0^+} x\ln x = 0\cdot\infty$ which is also an indeterminate form, but you cannot use L'Hopital's rule here. We must rewrite this expression to get one of the forms where L'Hopital's rule can be used.

$$\lim_{x\to 0^+} x\cdot\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\ln 0}{\frac{1}{0}} = \frac{\infty}{\infty}$$ now we can apply L'Hopital's rule.

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d(x^{-1})}{dx}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1x^{-2}} = \lim_{x\to 0^+} x^{-1}\cdot(-x^2) = \lim_{x\to 0^+}(-x) = 0$$

2 $\lim\limits_{x\to 0^+} u^v = 0^0, \infty^0, 1^\infty$, if you get one of these indeterminate forms use the steps below to find the limit.

Step 1: Let $y = u^v$

Step 2: Take the natural log of both sides of the equation and simplify the log expression

$$\ln y = \ln u^v$$

$$\ln y = v\cdot\ln u$$

Step 3 Take the limit of both sides

$$\lim_{x\to c} \ln y = \lim_{x\to c} v\cdot\ln u$$

Step 4: Solve for y by taking e of both sides

Ex: $\lim\limits_{x\to 0} x^x$

Step 1: Let $y = x^x$

Step 2: Take the *ln* of both sides of the equation and simplify the log expression

$$\ln y = \ln x^x$$

$$\ln y = x \cdot \ln x$$

Step 3 Take the limit of both sides (using the rule above) and simplify.

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} 0 \cdot \ln 0$$

$$\ln y = 0 \cdot \infty$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}}$$

$$\ln y = \lim_{x \to 0} \frac{\frac{1}{x}}{-x^{-2}}$$

$$\ln y = \lim_{x \to 0} (-x)$$

$$\ln y = 0$$

Step 4: Solve for y by taking e of both sides

$$e^{\ln y} = e^0$$

$$y = 1$$

Since we let

$$y = x^x$$

$$\lim_{x \to 0} x^x = 1$$

Ex 2: $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^{8x} = \lim_{x \to 0} \left(1 + \frac{1}{0}\right)^0 = \infty^0$

Make the following substitution: $y = \left(1 + \frac{1}{x}\right)^{8x}$

$$\ln y = \ln\left(1 + \frac{1}{x}\right)^{8x}$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0} 8x \cdot \ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\cdot \ln\frac{\left(1+\frac{1}{x}\right)}{\frac{1}{8x}} = \frac{\infty}{\infty}$$

Simplify and apply L'Hopital's rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\cdot\left(\frac{-x^{-2}}{-\frac{x^{-2}}{8}}\right)$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\cdot 8$$

$$\ln y = 8$$

$$e^{\ln y} = e^{8}$$

$$y = e^{8}$$

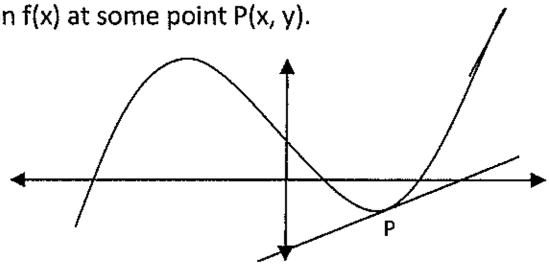
Therefore, $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^{8x} = e^{8}$

B. Derivatives

The derivatives of a function can be written in several forms, we will give three that are used more often.

a. Given a function: $f(x)$, the derivative can be written as $f'(x)$.
b. Given: y, the derivative can be written as y'.
c. Given: y, the derivative can be written as $\frac{d(y)}{dx}$ or $\frac{dy}{dx}$. .

The Derivative (the slope of a tangent line) helps us to find the equation of the tangent line of some function f(x) at some point P(x, y).

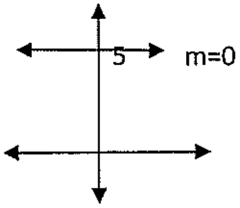


Power Rule

Rule 1: Derivative of a constant: $\frac{d(a)}{dx}=0$, **where *a* is any constant. The slope of a constant function m = 0.**

Ex 1: $\frac{d(5)}{dx}=0$

Ex 2: $\frac{d(\pi)}{dx}=0$



Ex 3: If $y=5^{\pi}$, then $y'=0$

Ex 4: If $f(x)=5^{\pi}+8$, then $f'(x)=0$

Note: Keep in mind that π =3.14 and e=2.718 are constants.

Rule 2: Power rule: The power rule should be used when taking the derivative of a function in the form of: $y = ax^p$, where a and p are constants.

Power Rule: Given: $y = ax^p$

$$y' = p \cdot ax^{p-1}$$

Ex 1: $y = x^4$

$$\frac{d(x^4)}{dx} = 4 \cdot x^{4-1}$$

$$= 4x^3$$

Ex 2: $y = 5x^7$

$$y' = 7 \cdot 5x^6$$

$$= 35x^6$$

When taking the derivative of terms being added or subtracts, take the derivative of each term separately.

Ex 3: $f(x) = 3x^2 + 2x - 1$

$$f'(x) = 6x + 2$$

Exponential function

Rule 3: rule: $\frac{d(e^u)}{dx} = u' \cdot e^u$

Ex 1: Find the derivative of:

$$y = e^{2x}$$

$$y' = 2 \cdot e^{2x}$$

Ex 2 : $\frac{d(e^{(4x^3+3x+4)})}{dx} = u' \cdot e^u$ Let $u = 4x^3 + 3x + 4$ and $u' = (12x^2 + 3)$

Now use Rule 3 above.

$$= (12x^2 + 3) \cdot e^{(4x^3+3x+4)}$$

Ex 3: Find the derivative

$$f(x) = x^4 + e^{3x} + \pi$$

$$f'(x) = 4x^3 + 3e^{3x}$$

Natural Log Rule

Rule 4: Derivative of a natural log function: $\dfrac{d(\ln u)}{dx} = \dfrac{u'}{u}$

Ex 1:

$\dfrac{d(\ln x^3)}{dx} = \dfrac{u'}{u}$ $\quad$ $\mathrm{u} = \mathrm{x}^3$ and $\mathrm{u}' = 3\mathrm{x}^2$

$= \dfrac{3\mathrm{x}^2}{\mathrm{x}^3}$ $\quad$ Simplify

$= \dfrac{3}{\mathrm{x}}$

Ex 2:

$f(x) = \ln \sin x$ $\quad$ Let $\mathrm{u} = \sin \mathrm{x}$ and $\mathrm{u}' = \cos \mathrm{x}$

$f'(x) = \dfrac{\cos x}{\sin x}$

$= \cot \mathrm{x}$

Trig Function Rules

Rule 4 Derivative of Trig functions

When finding the derivative of a trig function let the angle of the trig function *represent u. All trig functions must have an angle, the angle is always the factor that follows the trig function.* It will be necessary for students to learn the rules below.

a. $y = \sin u$
$y' = u' \cdot \cos u$

b. $y = \cos u$
$y' = u' \cdot (-\sin u)$

c. $y = \tan u$
$y' = u' \cdot \sec^2 u$

d. $y = \cot u$
$y' = u' \cdot (-\csc^2 u)$

e. $$y = \sec u \qquad y' = u' \cdot \sec u \cdot \tan u$$

f. $$y = \csc u \qquad y' = u' \cdot (-\csc u \cdot \cot u)$$

Ex 1:

$$y = \sec(3x^7) \qquad \mathrm{u} = 3\mathrm{x}^7 \qquad \mathrm{u}' = 21\mathrm{x}^6$$

$$y' = 21x^6 \cdot \sec 3\mathrm{x}^7 \cdot \tan 3\mathrm{x}^7$$

Ex 2:

$$y = \sin e^{2x} \qquad \mathrm{u} = \mathrm{e}^{2\mathrm{x}} \qquad \mathrm{u}' = 2\mathrm{e}^{2\mathrm{x}}$$

$$y' = 2\mathrm{e}^{2\mathrm{x}} \cdot \cos \mathrm{e}^{2\mathrm{x}}$$

Product Rule

Rule 2: Product rule: The product rule should be used if two factors are being multiplied. Given the product of two factors

$$y = u \cdot v$$

$$\frac{d(u \cdot v)}{dx} = u' \cdot v + u \cdot v'$$

The rule states that the derivative of the first factor multiplied by the second factor plus the first factor multiplied by the derivative of the second factor.

Ex 1:

$$\frac{d\left[x^3(2x^3 - 2)\right]}{dx} = \qquad Let\ u = x^3 \quad and \quad v = (2x^3 - 2)$$

$$u' = 3x^2 \quad \text{and} \quad v' = 6x^2$$

$$\frac{d(u \cdot v)}{dx} = 3x^2 \cdot (2x^3 - 2) + x^3 \cdot 6x^2$$

$$= 6x^5 - 6x^2 + 6x^5$$

$$= 12x^5 - 6x^2$$

$$= 6\mathrm{x}^2(2x^3 - 1)$$

Ex 2: Find the derivative

$$\frac{d\left[x^3 \cdot e^{-3x}\right]}{dx} =$$

$Let\ u = x^3 \quad and \quad v = e^{-3x}$

$u' = 3x^2$ and $v' = -3e^{-3x}$

$$\frac{d(u \cdot v)}{dx} = 3x^2 \cdot e^{-3x} + x^3(-3e^{-3x})$$

$$= 3x^2e^{-3x} - 3x^3e^{-3x}$$

Factor to simplify

$$= 3x^2e^{-3x}(1-x)$$

Ex 3:

$$\frac{d\left[e^{-3x} \cdot \sin 3x\right]}{dx} =$$

$Let\ u = e^{-3x} \quad and \quad v = \sin 3x$

$u' = -3e^{-3x}$ and $v' = 3\cos 3x$

$$\frac{d(u \cdot v)}{dx} = -3e^{-3x}\sin 3x + e^{-3x}(3\cos 3x)$$

$$= -3e^{-3x}\sin 3x + 3e^{-3x}\cos 3x$$

$$= -3e^{-3x}(\sin 3x - \cos 3x)$$

Quotient Rule

Rule 3: Quotient rule: The quotient rule should be used when taking the derivative of an algebraic fraction. Given

$$y = \frac{u}{v}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Ex1 :

$$\frac{d\left[\frac{x^3}{\left[(2x^3-2)\right]}\right]}{dx}=$$

$Let\ u=x^3 \quad and \quad v=(2x^3-2)$

$u'=3x^2$ anud $v'=6x^2$

$$\frac{d\left(\frac{\mathrm{u}}{\mathrm{v}}\right)}{dx}=\frac{3x^2\cdot(2x^3-2)-x^3\cdot 6x^2}{(2x^3-2)^2}=\frac{6x^5-6x^2-6x^5}{(2x^3-2)^2}=\frac{-6x^2}{(2x^3-2)^2}$$

Ex 2:

$$\frac{d\left[\frac{e^{5x}}{\tan 2x}\right]}{dx}=$$

$Let\ u=e^{5x} \quad and \quad v=\tan 2x$

$u'=5\mathrm{e}^{5\mathrm{x}}$ and $v'=2\sec^2 2x$

$$=\frac{5\mathrm{e}^{5\mathrm{x}}\tan 2x-e^{5x}\left(2\sec^2 2x\right)}{(\tan 2x)^2}$$

$$=\frac{\mathrm{e}^{5\mathrm{x}}(5\tan 2x-2\sec^2 2x)}{(\tan 2x)^2}$$

Chain Rule

Rule5: The Chain rule: The chain rule should be used when the entire expression is raised to a power.

$$y=[f(x)]^p$$
$$y'=p\cdot[f(x)]^{p-1}\cdot f'(x)$$

Ex 1: Use the chain rule to find the derivative

$$y = [(3x^2 - 5)]^5 \qquad\qquad \text{Note}: f(x) = 3x^2 - 5 \quad \text{and} \quad f'(x) = 6x$$

$$y' = 5 \cdot [(3x^2 - 2)]^{5-1} \cdot 6x$$

$$y' = 30x(3x^2 - 2)^4$$

Ex 2: Use the chain rule to find the derivative

$$y = \tan^3(3x^4 + e^{3x})$$

The above equation can be written as below and the chain rule can be used.

$$y = [\ \tan(3x^4 + e^{3x})\ ]^3$$

$$y' = 3[\ \tan(3x^4 + e^{3x})\ ]^{3-1} \sec^2(3x^4 + e^{3x})(12x^3 + 3e^{3x})$$

$$y' = 3(12x^3 + 3e^{3x})[\tan(3x^4 + e^{3x})]^2 \sec^2(3x^4 + e^{3x})$$

$$y' = 3(12x^3 + 3e^{3x})\tan^2(3x^4 + e^{3x})\sec^2(3x^4 + e^{3x})$$

Application of Derivatives

A. Tangent Line to a curve

To find the equation of a tangent line to a curve at a point (x, y).

1. Find the derivative of the function
2. Replace the x with the x-value in the given point, this is the slope of the tangent line.
3. Use the point slope formula to find the equation of the tangent line.

Example 1: Find the equation of the tangent line to the curve below at the point (2, 6)

$$y = 3x^2 + 2x - 4$$

$$y' = 6x + 2$$

$$m_{\tan} = 6(2) + 2$$

$$= 14$$

Use the point slope formula: $y - y_1 = m(x - x_1)$

$$y - 6 = 14(x-2)$$
$$y = 14x - 22$$

Higher order derivatives

The notation $\frac{d^2y}{dx^2}$ indicates the **second** derivative of some function y with respect to x. To do so find the first derivative, simplify the problem, then take the derivative again. The second derivative can also be indicated as following: $f''(x)$ and y''.

Find the second derivative of the expressions below:

Example 1: $y = 4x^3 + 23x + \ln x$

$$y' = 12x^2 + 23 + \frac{1}{x}$$
$$y' = 12x^2 + 23 + x^{-1}$$
$$y'' = 24x - x^{-2}$$

Example 2: $y = e^{3x^2} + x^3$

$$y' = 6xe^{3x^2} + 3x^2$$

$$y'' = 6 \cdot e^{3x^2} + 6x \cdot 6xe^{3x^2} + 6x$$
$$= 6e^{3x^2} + 36x^2e^{3x^2} + 6x$$

Note: The produce rule was used on the first term.

Example 3: $y = 3x(\sqrt{x^2 + 2x})$

When finding derivatives and a radical is involve change it to a fractional exponent first the proceed with the derivative.

$$y = 3x(x^2 + 2x)^{\frac{1}{2}}$$

$$y' = 3(x^2 + 2x)^{\frac{1}{2}} + 3x\left[\frac{1}{2}\left(x^2 + 2x\right)^{-\frac{1}{2}}\right](2x + 2)$$

Factor out $3\left(x^2 + 2x\right)^{-\frac{1}{2}}$

$$y' = 3\left(x^2 + 2x\right)^{-\frac{1}{2}}\left[\left(x^2 + 2x\right) + \frac{x}{2}(2x + 2)\right]$$

$$y' = 3\left(x^2 + 2x\right)^{-\frac{1}{2}}\left[\left(x^2 + 2x\right) + x^2 + x\right]$$

$$y' = 3(x^2+2x)^{-\frac{1}{2}}[2x^2+3x]$$

$$y'' = 3\left[-\frac{1}{2}(x^2+2x)^{-\frac{3}{2}}(2x+2)(2x^2+3x)+(x^2+2x)^{-\frac{1}{2}}(4x+3)\right]$$

$$y'' = 3(x^2+2x)^{-\frac{3}{2}}[-(x+1)(2x^2+3x)+(x^2+2x)(4x+3)]$$

$$y'' = 3(x^2+2x)^{-\frac{3}{2}}[-2x^3-3x^2-2x^2-3x+4x^3+11x^2+6x]$$
$$= 3(x^2+2x)^{-\frac{3}{2}}[2x^3+6x^2+3x]$$

Velocity and Acceleration

If we let s(t) be the coordinate at time t of a point P on a coordinate line l, then

1. The velocity of P is $v(t)=s'(t)$
2. The acceleration of P is $a(t)=v'(t)=s''(t)$

Example 1: Find the velocity and acceleration at time t, given: $s(t)=2t^4-6t^2$

Using 1 above we can find the velocity by taking the derivative of s(t): $v(t)=s'(t)$

$$s(t)=2t^4-6t^2$$

$$v(t)=s'(t)=8t^3-12t$$

Since the acceleration equal: $a(t)=v'(t)=s''(t)$, then

$$a(t)=v'(t)=s''(t)=24t^2-12$$

Implicit differentiation

Use implicit differentiation if you cannot solve the equation for one of the variable in terms of the other, that is , if you can't rewrite your equation in one of the forms below:

Given two variables x and y:

a. y= f(x) or b. x=f(y)

Using Implicit differentiation you take the derivative of the term but then multiply it by it's derivative w.r.t. some variable.

Step 1: Determine which variable you are taking the derivative *with respect to* (w. r.t), use this notation when you are taking the derivative with respect to x.

Ex: A. $\frac{d(?)}{dx} \begin{matrix}\rightarrow \\ \overline{\quad} \\ \rightarrow\end{matrix} \frac{\text{variable you are taking the derivative of}}{\text{variable with respect to}}$

Step 2:. Take the derivative of each term the normal way, but you must include a factor as given above with each term.

Ex 1: Solve for $\frac{d(y)}{dx}$. If I am take the derivative of the expression below *with respect to* x

$$x^3 + 3y^4 = 2$$

We take the derivative as in explicit derivatives, but all terms must include a factor as in Ex: A above.

$$\underset{\substack{\uparrow \\ x-term-\text{w.r.t. } x}}{3x^2\frac{d(x)}{dx}} \quad + \quad \underset{\substack{\uparrow \\ y-term-\text{w.r.t } x}}{12y^3\frac{d(y)}{dx}} = 0 \cdot \frac{d(\text{constant})}{dx}$$

Step 3: Simplify as much as possible. Note: Let $\frac{d(x)}{dx}$=1

$$3x^2 \cdot 1 \quad + \quad 12y^3\frac{d(y)}{dx} = 0 \cdot$$

Step 4: Solve for $\frac{d(y)}{dx}$ as you would in solving a linear equation.

$$12y^3\frac{d(y)}{dx} = -3x^2$$

$$\frac{d(y)}{dx} = \frac{-3x^2}{12y^3}$$

$$= \frac{-x^2}{4y^3}$$

Ex 2: Find $\frac{d(y)}{dx}$, given: $x^2+3xy+4y^2=3$ Note: 3xy is a product so we use the product rule.

$$2x\frac{d(x)}{dx}+3\left[1\frac{dx}{dx}\cdot y+x\cdot 1\frac{dy}{dx}\right]+8y\frac{dy}{dx}=0$$

$$2x+3y+3x\frac{dy}{dx}+8y\frac{dy}{dx}=0$$

$$3x\frac{dy}{dx}+8y\frac{dy}{dx}=-(2x+3y)$$

$$(3x+8y)\frac{dy}{dx}=-(2x+3y)$$

$$\frac{dy}{dx}=\frac{-(2x+3y)}{(3x+8y)}$$

Related Rates Problems: Use implicit differentiation

Problem1: If $5x^2+xy+y^2=7$, and $\frac{dy}{dt}=3$, find $\frac{dx}{dt}$ when x =1 and y =2.

$$5x^2+xy+y^2=7$$

$$10x\frac{dx}{dt}+1\cdot\frac{dx}{dt}y+x\cdot 1\frac{dy}{dt}+2y\frac{dy}{dt}=0$$

$$10x\frac{dx}{dt}+y\frac{dx}{dt}+x\frac{dy}{dt}+2y\frac{dy}{dt}=0$$

Now plug in the given values

$$10(1)\frac{dx}{dt}+(2)\frac{dx}{dt}+(1)(3)+2(2)(3)=0$$

$$10\frac{dx}{dt}+2\frac{dx}{dt}+3+12=0$$

$$12\frac{dx}{dt}+15=0$$

$$\frac{dx}{dt} = -\frac{15}{12} = -\frac{5}{4}$$

Second Derivatives

Ex: Find $\frac{d^2y}{dx^2}$: $x^{-2} + e^y = 7$

$$-2x^{-3}\frac{dx}{dx} + e^y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x^{-3}}{e^y}$$

$$\frac{dy}{dx} = 2x^{-3}e^{-y} \qquad \text{Eq } *$$

$$\frac{d^2y}{dx^2} = -6x^{-4}\frac{dx}{dx}e^{-y} - 2x^{-3}e^{-y}\frac{dy}{dx} \qquad \text{Eq } **$$

We replace $\frac{\mathrm{dy}}{\mathrm{dx}}$ in Eq ** by $\frac{\mathrm{dy}}{\mathrm{dx}}$ in Eq *

$$\begin{aligned}\frac{d^2y}{dx^2} &= -6\mathrm{x}^{-4}\mathrm{e}^{-\mathrm{y}} - 2\mathrm{x}^{-3}\mathrm{e}^{-\mathrm{y}}(2\mathrm{x}^{-3}\mathrm{e}^{-\mathrm{y}}) \\ &= -6x^{-4}e^{-y} - 4x^{-6}e^{-2y} \\ &= -2x^{-6}e^{-2y}\left(3x^2e^y + 2\right)\end{aligned}$$

Example 2: Find the equation of the tangent line to the curve below at the point (2, -1), given $3x^2 - 2y^2 + 5y = 5$

Step 1: Find $\frac{dy}{dx}$ (Hint: Use implicit differentiation)

$$3x^2 - 2y^2 + 5y = 5$$

$$6x\frac{dx}{dx} - 4y\frac{dy}{dx} + 5\frac{dy}{dx} = 0$$

$$6x - 4y\frac{dy}{dx} + 5\frac{dy}{dx} = 0$$

$$(-4y+5)\frac{dy}{dx}=-6x$$

$$\frac{dy}{dx}=\frac{-6x}{(-4y+5)}$$ Now let x=2 and y =-1

$$\frac{dy}{dx}=\frac{-6(2)}{(-4(-1)+5)}$$

$$\frac{dy}{dx}=\frac{-12}{9}$$

$$m=\frac{-12}{9}=-\frac{4}{3}$$

Use the point slope formula: $y-y_1=m(x-x_1)$

$$y+1=-\frac{4}{3}(x-2)$$

$$y+1=-\frac{4}{3}(x-2)$$ Multiply by 3.

$$3y+3=-4x+8$$

$$4x+3y=5$$ Equation of the tangent line.

C. Integration

This section discuss two types of integration: a) indefinite and b) definite integration. Integration is the reverse of differentiation which is sometimes referred as antiderivatives (indefinite integral). Definite integral defines the area under a curve on the interval [a, b]. We will treat indefinite integrals first.

Indefinite integral

Constant rule

Rule 1: $\int a \; dx = ax + C$ where a is a constant.

Example 1: $\int 4 \; dx = 4x + C$ =

Power rule

Rule 2: The power rule should be used for integral of the form: $\int ax^{p} dx$, where a is a constant.

Power rule: $\int ax^{p} dx = a \cdot \frac{x^{p+1}}{p+1} + C$

Exampl1: $\int x^{4} dx = \frac{x^{4+1}}{4+1} + C$

$$= \frac{x^{5}}{5} + C$$

Example2: $\int \sqrt[3]{x} dx = \int x^{\frac{1}{3}} dx$

$$= \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{x^{\frac{1}{3}+\frac{3}{3}}}{\frac{1}{3}+\frac{3}{3}} + C$$

$$= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$= \frac{3x^{\frac{4}{3}}}{4} + C$$

$$\text{Example 3: } \int 5x^7\,dx = 5\int x^7\,dx$$
$$= 5\cdot\frac{x^{7+1}}{7+1}+C$$
$$= \frac{5\text{x}^8}{8}+C$$

$$\text{Example 4: } \int (x^{-3}+2x-500)dx = -\frac{\text{x}^{-2}}{2}+x^2-500x+C$$

Definite Integral

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a,b]$ and if F(x) is an antiderivative of $f(x)$ on $[a,b]$ then

$$\int_a^b f(x)dx = F(b)-F(a)$$

This theorem gives us to important results:

Fact 1. It helps us to evaluate a definite integral of a continuous on a closed interval, that is,

$$F(x)\Big|_a^b = F(b)-F(a)$$

Fact 2. Since $\int_0^x f(t)dt = F(x)-F(0)$ taking the derivative with respect to x we get

$$\frac{d}{dx}\int_0^x f(t)dt = F(x) \qquad \text{But F(x)} = f(x) \quad \textit{therefore} \qquad \frac{d}{dx}\int_0^x f(t)dt = f(x)$$

Examples of Fact 1

Fact 1: Ex 1: $\int_0^3 4x^3dx = x^4 \Big|_0^3 = (3)^4-(0)^4 = 81$

Ex 2: Find the area under the curve $y = x^2-8$ on the interval [1, 5].

$$\int_1^5 (x^2-8)\,dx = (\frac{1}{3}x^3-8x)\Big|_1^5 = \left[\frac{1}{3}(5)^3-8(5)\right]-\left[\frac{1}{3}(1)^3-8(1)\right]$$

$$= \frac{125}{3}-40-\frac{1}{3}+8$$

$$= \frac{28}{3}$$

Examples of Fact 2

Fact 2: (Long verison) $\frac{d}{dx}\int_0^x 4t^3dt = \frac{d}{dx}(t^4)\Big|_0^x = \frac{d(x^4)}{dx} = 4x^3$ which is the original function.

Using the Fact 2: $\frac{d}{dx}\int_0^x 4t^3dt = 4x^3$

U-Substitution

Many integrals can be evaluated by using U-Substitution. Below are steps that can be use for this method.

Step 1: Determine the substitution for u and find the derivative of u w.r.t the variable you are integrating (we will call it dx).

Step 2: Solve for dx

Step 3: Substitute for u and dx and simplify. Your problem should only have u-terms.

Step 4: Simplify, if you have followed the steps correctly your problem should be in a form of one of the basic rules for integrating.

Step 5: Integrate

Step 6: Replace u with your original substitution and add + C.

U-substitution will reduce a more complex expression to a form that can easily be integrated. The basic forms are given below.

Rule 1: Power rule (with u-substitution)

1. $\int [f(u)]^p du = \frac{1}{p+1}[f(u)]^{p+1} + C$

Rule 2: **Exponential rule** – Use this rule when your integral has only one exponential factor, letting the exponent of the e represent *u*.

2. $\int e^u du = e^u + C$

Rule 3: Natural Log – these integral results is a natural log function.

$$\int \frac{du}{u} = \ln|u| + C$$

Examples of U-Substitution

Rule 1: $\int (u)^p du = \frac{1}{p+1}(u)^{p+1} + C$ Where u is some function of $f(x)$

Example 1:

$$\int_2^{10} \frac{1}{\sqrt{5x-1}} dx = \int_2^{10} \frac{1}{(5x-1)^{\frac{1}{2}}} dx$$

$$= \int_2^{10} (5x\text{-}1)^{-\frac{1}{2}} dx$$

If we let u=5x-1 du=5dx

$$= \int_2^{10} u^{-\frac{1}{2}} \frac{du}{5}$$

$\frac{du}{5}$=dx

$$= \frac{1}{5}\int_2^{10} u^{-\frac{1}{2}} du$$

We now can use the power rule and evaluate.

$$\frac{1}{5} \cdot 2u^{\frac{1}{2}} = \frac{2}{5}(5x-1)^{\frac{1}{2}}\Big|_2^{10}$$

$$= \left[\frac{2}{5}(5(10)-1)^{\frac{1}{2}}\right] - \left[\frac{2}{5}(5(2)-1)^{\frac{1}{2}}\right]$$

$$= \left[\frac{2}{5}(49)^{\frac{1}{2}}\right] - \left[\frac{2}{5}(9)^{\frac{1}{2}}\right]$$

$$= \frac{14}{5} - \frac{6}{5} = \frac{8}{5}$$

Rule 2: $\int e^{u}du = e^{u} + C$

Example 2: $\int e^{2x}\ dx =$

$\downarrow \ \downarrow$

Let $u = 2x$

$du = 2\,dx$

$\frac{du}{2} = dx$

$\int e^{u}\ \frac{du}{2} =$ Make the substitutions in the integral

$\frac{1}{2}\int e^{u}\ du = \frac{1}{2}e^{u} + C$ Factor out the constant we get the form of the rule so we can integrate

$= \frac{1}{2}e^{2x} + C$ Replace u with 2x

Example 3: $\int \cos x\ e^{\sin x}\ dx =$

Let $u = \sin x$

$du = \cos x\ dx$

$\frac{du}{\cos x} = dx$

$\int \cos x\ e^{u}\ \frac{du}{\cos x} =$ Make the substitutes in the integral

$\int e^{u}\ du = e^{u} + C$ Simplifying we get the form of the rule so we can integrate

$= e^{\sin x} + C$ Let $u = \sin x$

Example 4: $\int 3x^{2}\sqrt{x^{3}+1}\ dx =$

Let $u = x^{3} + 1$

$du = 3x^{2}\ dx$

$\frac{du}{3x^{2}} = dx$

$\int 3x^{2}\sqrt{u}\ \cdot \frac{du}{3x^{2}}$

Simplifying the integral we get

$$\int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left(x^{3}+1\right)^{\frac{3}{2}} + C$$

Integrals Resulting in a Natural Log Function.

When you have a fractional expression where the factor in your denominator is raised to the first power and none of the basic rules applies, try using this method. Let your denominator equal u and find du to determine if your problem can reduce to the form below.

$$Rule\,3: \quad \int \frac{du}{u} = \ln|u| + C$$

$$\text{Example 1: } \int \frac{3\text{dx}}{\text{x}}$$

Let u = x

du = dx

Make the substitutions

$$\text{Example 1: } \quad 3\int \frac{\text{du}}{\text{u}} = 3\ln|u| + C$$

$$= 3\ln|\text{x}| + C$$

$$\text{Example 2: } \quad \int \frac{\cos \text{x}}{\sin \text{x}} dx =$$

$$\text{Let } \text{u} = \sin \text{x}$$

$$\text{du} = \cos \text{x dx}$$

$$\frac{\text{du}}{\cos \text{x}} = dx$$

$$\int \frac{\cos \text{x}}{\text{u}} \cdot \frac{\text{du}}{\cos \text{x}} =$$

$$\int \frac{\text{du}}{u} = \ln|u| + C$$

$$= \ln|\sin \text{x}| + C$$

$$\text{Example 3: } \quad \int \frac{6\text{x}}{\text{x}^2 + 8} dx =$$

$$\text{Let } \text{u} = (\text{x}^2 + 8) \qquad \text{this factor is of power 1.}$$

$$\text{du} = 2\text{x dx}$$

$$\frac{\text{du}}{2\text{x}} = dx$$

$$\int \frac{6\text{x}}{\text{u}} \cdot \frac{\text{du}}{2\text{x}} =$$

$$\int \frac{3}{\text{u}} \cdot \frac{\text{du}}{} =$$

$$3\int \frac{\text{du}}{u} = 3\ln|u| + C$$

$$= 3\ln|x^2 + 8| + C$$

Example 4 : $\int \cot x \ dx =$

This is not one of the rule but we can change $\cot x = \frac{\cos x}{\sin x}$

Rule 4 : $\int \frac{\cos x}{\sin x} \ du =$

Let $u = \sin x$

$du = \cos x \, dx$

$$\int \frac{\cos x}{u} \cdot \frac{du}{\cos x} =$$

$$\frac{du}{\cos x} = dx$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sin x| + C$$

Basic Integral of Trig Functions

Below are the rules for integral of basic trig functions students should know. Use u-substitutions to try to get your integral in one of the forms below. Please review your trig identities; you may have to change the form of the problems to get one of the forms below.

Rule 1 : $\int \sin u \ du = -\cos u + C$

Rule 2 : $\int \cos u \ du = \sin u \ + C$

Rule 3 : $\int \sec^2 u \ du = \tan \ u + C$

Rule 4 : $\int \sec u \tan u \ du = \sec u \ + C$

Example 1:

$$\int \sin 4x \ dx \qquad\qquad \text{Step 1: Let u = 4x}$$

$$du = 4\,dx$$

$$\frac{du}{4} = dx$$

$$\frac{1}{4}\int \sin\ u\ \ du = \frac{1}{4}(\text{-cos}\ \ u) + C$$

$$= -\frac{1}{4}\ \cos\ \ 4x + C$$

$$\textit{Example 2}: \quad \int \frac{1}{\sec x}\ dx$$

We see this is not one of the basic rules, but we can change the form, let $\frac{1}{\sec x} = \cos x$

$$\int \frac{1}{\sec x} dx = \int \cos x\, dx \qquad\qquad u = x \ \text{ and } \quad du = dx$$

$$= \int \cos u\, du$$

$$= \sin u + C$$

$$= \sin x + C$$

Example 3:

$$\int (3x^2 + 3)\sin(x^3 + 3x)dx = \qquad\qquad u = x^3 + 3x \qquad du = (3x^2 + 3)dx$$

$$\int (3x^2 + 3)\sin u \frac{du}{(3x^2 + 3)} = \qquad\qquad \frac{du}{(3x^2 + 3)} = dx$$

$$\int \sin u \ \ du = -\cos u + C$$

$$= \text{-}\cos(x^3 + 3x) + C$$

Techniques of integration

If none of the above methods works then you may have to use one of the many techniques of integration. This review will present two techniques that are used most often in differential equation. The two methods are: Integration by parts and Integration by partial fractions.

Technique I: Integral By Parts

A. Integration by parts (BP) is somewhat like the product rules of antiderivatives. If we use the product rule with integrals instead of derivative we can get the rule for integration by parts.

$$u \cdot v = \int u \cdot dv + \int v \cdot du$$

Rewriting this equation we get the rule below which we use in Calculus II:

$$BP\,\text{rule}: \quad \int u \cdot dv = uv - \int v \cdot du$$

B. When should you use integration by part?

1. If you have at least two factor and none of the basic rules apply and

2. You can integrate at least one of the factors.

Steps for Integrating using the BP method

Given an integral, that you have determined that the BP method may applies, you should:

Step 1: Determine if at least one of the factors can be integrated (In many cases you may be able to integral both terms).

Step 2: Let the factor that, when you take the derivative, will reduce to a lower form represent *u*, provided that you can integrate the other factor. Let the other factor represent dv.

Step 3: Find the derivative of u and integrate the dv.

Step4: Substitute the values in the formula, simplify by trying to integrating the late part of the formula.

Step 5: If you cannot integrate the last part of the formula repeat the above steps on the new integral until you can integral the final integral.

$Example\,1: \ \int x \cdot e^{2x} dx$

Step 1: You can integrate both x and e^{2x}

Step 2: When you take the derivative of x (a linear function) it reduce to a constant (a lower level function). When you take the derivative of e^{2x} it does not reduce to a lower level function, therefore we let $u = x$ and $dv = e^{2x}$.

Example 1: $\int x \cdot e^{2x} dx$

Step 3:

$$Let: \quad \mathrm{u = x} \quad \text{and} \quad \mathrm{dv} = \int e^{2x} dx$$

$$\mathrm{du = 1\,dx} \quad \text{and} \quad \mathrm{v} = \frac{1}{2}\mathrm{e}^{2x}$$

Step 4: Substitute into the formula

$$\int u \cdot \; dv \;=\; u \cdot v \;-\int v \cdot du$$

$$\int x \cdot e^{2x} dx = x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \cdot 1dx$$

$$= \frac{\mathrm{xe}^{2x}}{2} - \frac{1}{2} \int e^{2x} dx$$

$$\text{Note}: \frac{1}{2} \int \mathrm{e}^{2x} \mathrm{dx} = \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C = \frac{1}{4} e^{2x} + C$$

$$= \frac{\mathrm{xe}^{2x}}{2} - \frac{1}{4} e^{2x} + C$$

Example 2: $\int \ln x \cdot dx$

Note: dx is consider a factor

Step 1: You can integrate dx, but not ln x

Step 2: We can skip this step.

Step 3:

$$Let: \quad u \;=\; \ln x \quad and \quad dv \;=\; \int dx$$

$$du \;=\; \frac{1}{x} dx \quad and \quad v \;=\; x$$

Step 4: Substitute in the formula

$$\int u \cdot \; dv \;=\; u \cdot v \;-\int v \cdot du$$

$$\int \ln x \cdot dx \;=\; \ln x \cdot (x) - \int x \cdot \frac{1}{x} dx$$

$$= \mathrm{x\ln x - x + C}$$

Technique II: Integration By Partial Fractions

In this section we will only discuss how to decompose the algebraic fraction that will factor into linear factors.

Use the partial fraction method if when given a rational expression:

Method I

a. The degree of the denominator is greater than the degree of the numerator and

b. The denominator can be factored.

Method II

a. To decompose fractions in which the degree of the numerator is greater than or equal to the degree of the denominator and

b. The denominator can be factored.

Method I

Step 1: Factor the denominator of your problem, if needed.

Step 2: Determine the number of simple fractions into which the problems can decomposed. If the denominator contains **n**-factors, then the problem can be rewritten as the sum of **n** new fractions.

Step 3: Set up the new decomposed fractions, letting each factor be a denominator. Let the numerators be some constant A, B, C, ...

Use the following steps to solve the problems.

R1: Find the LCD and multiply each term by the LCD. The LCD will always be the original problem factored denominator.

R2: Simplify by canceling all common factors.

R3: Multiply factors to eliminate all grouping symbols.

R4: Equate like terms to form a system of equation.

R5: Eliminate the ***x*** in all equations by dividing by the appropriate power of ***x***.

R6: Solve the system for A, B...

R6: Integrate each fraction.

Example 1: $\int \frac{-2x+9}{(x+3)(x-2)}\,dx$

Decompose **the fraction:** $\frac{-2x+9}{(x+3)(x-2)}$.

Step 1: Factor the denominator of your problem, if needed. In this problem the denominator is already in factored form.

$$\frac{-2x+9}{(x+3)(x-2)}$$

Step 2: Determine into how many simple fractions the problems will decompose. If there are <u>n</u>-factors, then the problem can be rewritten as the sum of <u>n</u> new fractions. There are 2 factors; therefore, this problem can be decomposed into two new fractions:

$$\frac{-2x+9}{(x+3)(x-2)} = \text{——} + \text{——}$$

Step 3: Set up the decomposed fractions. Each factor will be a denominator for the new fractions. The numerators will be some constant A, B, C...

$$\frac{-2x+9}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$$

We now need to solve for the constants A and B.

R1: Find the LCD and multiply each term by the LCD. The LCD will always be the original problem factored denominator.

The LCD is (x + 3) (x – 2)

$$\frac{-2x+9}{(x+3)(x-2)}\cdot(x+3)(x-2) = \frac{A}{(x+3)}\cdot(x+3)(x-2) + \frac{B}{(x-2)}\cdot(x+3)(x-2)$$

R2: Simplify by canceling we get :

$$-2x+9 = A\cdot(x-2)+B(x+3)$$

R3: Eliminate all grouping symbols by multiplying:

$$-2x+9 = Ax-2A+Bx+3B$$

R4: Equate like terms to form a system of equation:

$$-2x = Ax + Bx$$
$$9 = -2A + 3B$$

R5: Eliminate the ***x*** in all equations by dividing by the appropriate power of ***x***.

$$-2 = A + B$$
$$9 = -2A + 3B$$

R6: Solve the system for A and B. In this case, multiply the first equation by 2 and add It to the second equation.

$$2(\ -2 = A + B) \ \text{-----}\!\!\rightarrow \qquad -4 = -2A + 2B$$
$$9 = -2A + 3B \ \text{-----}\!\!\rightarrow \qquad \underline{9 = -2A + 3B}$$
$$5 = 5B$$
$$1 = B$$

Plug the value for B in one of the two equations to determine that A = -3. Therefore,

$$\int \frac{-2x+9}{(x+3)(x-2)}dx = \int \frac{-3}{(x+3)}dx + \int \frac{1}{(x-2)}dx$$

Now integrate each of the new fractions by one of the methods above in this section.

$$= -3Ln(x+3) + Ln(x-2) + C$$

Method II

When the **degree of the fraction's numerator is greater than or equal to the degree of the denominator,** long division is needed. Polynomial division (long division) reduces the problem to a form where one of the methods above can be used to solve it. You should review long division before attempting this type of problem.

Example

Decompose: $\frac{2x^2+7}{x^2+6x+9}$ The degree of the numerator and denominator are the same

Solution

In this problem, the degree of the numerator is equal the degree of the denominator, so long division needs to be used first.

$$\begin{array}{r l} & \quad 2 \\ x^2+6x+9 \big) & \overline{2x^2 \qquad\quad +7} \\ & \underline{-2x^2-12x-18} \\ & \qquad -12x-11 \end{array} \quad \rightarrow \quad 2 + \frac{-12x-11}{x^2+6x+9}$$

The original problem now can be rewritten as follows:

$$2 + \frac{-12x-11}{x^2+6x+9}$$

In the fractional part the degree of the numerator is less than the denominator so we can use one of the previous methods to decompose the fractional part only:

$$\frac{-12x-11}{x^2+6x+9} = \frac{-12x-11}{(x+3)^2}$$

$$\frac{-12x-11}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$$

$$-12x-11 = A(x+3)+B$$

$$-12x-11 = Ax+3A+B$$

$$\begin{array}{lcl} -12x = Ax & \rightarrow & -12 = A \\ -11 = +3A+B & \rightarrow & -11 = 3(-12)+B \\ & & -11 = -36+B \\ & & 25 = B \end{array}$$

So,

$$\frac{-12x-11}{(x+3)^2} = \frac{-12}{(x+3)} + \frac{25}{(x+3)^2}$$ and since

$$2+\frac{-12x-11}{(x+3)^2} = \quad 2+\frac{-12}{(x+3)} + \frac{25}{(x+3)^2},$$ then

Going back to the original problem it has been decomposed to the below expression, now we can integrate using a previous method.

$$\frac{2x^2+7}{x^2+6x+9} = 2+\frac{-12x-11}{(x+3)^2}$$

$$= 2+\frac{-12}{(x+3)}+\frac{25}{(x+3)^2}$$

$$\int \frac{2x^2+7}{x^2+6x+9}\,dx = \int 2\,dx + \int \frac{-12}{(x+3)}dx + 25\int (x+3)^{-3}\,dx$$

$$= 2x - 12\ln|x+3| + 25\frac{(x+3)^{-2}}{-2} + C$$

Applied problems

Problem1: Given the velocity function v(t) at some t, find the position function s(t).

$$v(t) = 3x^5 + e^{3x}$$

From the section on higher order derivatives we know that

$$v(t) = s'(t)$$

Therefore we can take the antiderivative of v(t) to find s(t).

$$v(t) = 3x^5 + e^{3x}$$

$$\int v(t)dt = \int (3x^5 + e^{3t})dt$$

$$s(t) = \frac{1}{2}x^6 + \frac{1}{3}e^{3x} + C$$

Problem 2: Find f(x) subject to the given condition, where:

$f'(x) = 9x^2 + x - 8$ f(-1)=1

We integrate

$$f'(x) = 9x^2 + x - 8$$

$$\int f'(x)\,dx = \int (9x^2 + x - 8)dx$$

$$f(x) = 3x^3 + \tfrac{1}{2}x^2 - 8x + C$$

To find the original function we must solve for C. To do so we use the condition f (-1) = -1 by replacing -1 for x and -1 for f(-1) in the function.

$f(x) = 3x^3 + \frac{1}{2}x^2 - 8x + C$

$f(-1) = 3(-1)^3 + \frac{1}{2}(-1)^2 - 8(-1) + C$

$-1 = -3 + \frac{1}{2} + 8 + C$ Solve for C

$-\frac{13}{2} = C$

Using this value for C, we get: $f(x) = 3x^3 + \frac{1}{2}x^2 - 8x - \frac{13}{2}$

This type of problem is what is known as a differential equation. An equation that involves one or more derivatives is known as a differential equation.

Problem 3: Solve the differential equation below for f(x) subject to the given conditions:

$f''(x) = 5\cos x + 2\sin x$ subject to: $f(0) = 3$, $f'(0)=4$

Since the second derivative is given we will have to integrate twice.

$\int f''(x)dx = \int (5\cos x + 2\sin x)dx$ subject to: $f(0) = 3$, $f'(0)=4$

$f'(x) = 5\sin x - 2\cos x + C$

Use the condition for the first derivative and solve for C.

$$\begin{aligned} f'(x) &= 5\sin x - 2\cos x + C \\ 4 &= 5\sin(0) - 2\cos(0) + C \\ 4 &= 0 - 2 + C \\ 6 &= C \end{aligned}$$

subject to: $f(0) = 3$, $f'(0)=4$

$f'(x) = 5\sin x - 2\cos x + 6$

Integrating the second time and solving for C, we will be able to get the desired function f(x).

$$\begin{aligned} \int f'(x)dx &= \int (5\sin x - 2\cos x + 6)dx \\ f(x) &= -5\cos x - 2\sin x + 6x + C \\ 3 &= -5\cos(0) - 2\sin(0) + 6(0) + C \\ 8 &= C \end{aligned}$$

therefore, $f(x) = -5\cos x - 2\sin x + 6x + 8$

Example 4: Given: $dx = 5t^4\,dt$; if x = 3 when t = 2, **what is the value of x, when t = 1?**

$$\int dx = \int 5t^4 dt$$
$$x = t^5 + C$$

Using the conditions above

$$x = t^5 + C$$
$$3 = (2)^5 + C$$
$$3 = 32 + C$$
$$-29 = C$$

Therefore,

$$x = t^5 - 29 \qquad \text{let } t = 1$$
$$x = (1)^5 - 29$$
$$x = -28$$

Review

I. Find the Limit of the following:

1. $\lim_{x\to 0}\frac{x^2+4x-5}{x^2+x-2}$

2. $\lim_{x\to 2}\frac{x-2}{x^3-8}$

3. $\lim_{x\to 5^-}\frac{x^2-25}{x-5}$

4. $\lim_{x\to\infty}\frac{x^2-6x-3}{x^4+2x-1}$

5. $\lim_{x\to\infty}\frac{x^{\frac{1}{3}}}{x^4+2}$

6. $\lim_{x\to\infty}\frac{x+2}{e^x}$

7. $\lim_{x\to 0}\frac{\sin x}{2-2\cos x}$

8. $\lim_{x\to\infty}\frac{e^x}{x^2}$

II. Find the derivative of the following:

1. $y=e^{2x}+\ln(x-4)+\sec x$

2. $y=e^{-x}\sin e^{-x}$

3. $y=\ln(4x^2-3x-1)$

4. $y=\frac{3e^x}{x^{-4}}$

5. $y=\frac{1}{\sqrt{3x-4}}$

6. $y=\cos^5 4x$

7. $y=\sqrt{e^x}$

8. $y=\sin 3x^2$

9. $y=xe^{x^2}$

10. $y=\ln(\cos x)$

11. $y=x^4-6x+2$

12. $y=\sqrt{\pi^{5e}}$

13. $y=\sin 2x\sqrt{x^2+3}$

14. $y=(3x^3+4x-5)^{-3}$

III. Integrate

1. $\int e^{(3x+2)}dx$
2. $\int x^2 e^{x^3} dx$
3. $\int \tan x \; dx$
4. $\int \frac{1}{2x+7}dx$
5. $\int_1^4 \frac{1}{x^2}dx$
6. $\int \frac{\ln x}{x}dx$
7. $\int \frac{\sin x}{1+2\cos x} \; dx$
8. $\int \frac{2x+1}{\sqrt{x^2+x}} \; dx$
9. $\int \frac{x-1}{\left(x^2-2x+5\right)^3} \; dx$
10. $\int xe^{6x}dx$
11. $\int \frac{x+25}{x^2+2x-63} \; dx$
12. $\int x\cos x \; dx$
13. $\int \frac{x+18}{(x+5)(x-8)} \; dx$
14. $\int 3\ln x^5 dx$
15. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}} \; dx$

Applications

1. Find the equation of the tangent line to the curve below at the point P (2, 8).

$$y=\left(4x^2-8x+3\right)^4$$

2. Find the equation of the tangent line to the curve below at the point P (2, 1).

$$x^2+y^2-2x+3y=5$$

3. Find $\frac{dy}{dt}$ given: $xy=9$ subject to $\frac{dx}{dt}=2$ when $x=3$.

4. Find y''' given: $y=x^4+\sin x$

5. Solve the differential equation subject to the conditions given below.

$$\frac{d^2y}{dx^2}=6x-4 \qquad \text{subject to } y=4,\ y'=5 \text{ if } x=2.$$

6. Let v(t) represent the velocity at t, find the position s(t) and acceleration a(t) at time t.

Given $v(t)=3t^2+t^4+e^{3t}$

7. Given the determinant below:

   (a) Evaluate the determinant below using the cofactor method.

   (b) Find the cofactor for the value in the a_{21} position.

8. Find the equation of a line passing through the point (3,- 3) and parallel to the line y=7x+3.

9. Solve the system of equations below:

$$\begin{aligned} y^2 &= x \\ x+2y-3 &= 0 \end{aligned}$$

10. If

$\int_1^2 f(x)dx = -4$ and $\int_1^5 f(x)dx=6$ find $\int_2^5 f(x)dx$ given:

$$\int_1^2 f(x)dx + \int_2^5 f(x)dx = \int_1^5 f(x)dx$$

Solutions to the Review

Review

Find the limit

$$1.\ \lim_{x\to 0} \frac{x^2+4x-5}{x^2+x-2}$$

$$= \lim_{x\to 0} \frac{0^2+4(0)-5}{(0)^2+(0)-2} = \frac{-5}{-2} = \boxed{\frac{5}{2}}$$

$$2.\ \lim_{x\to 2} \frac{x-2}{x^3-8}$$

$$= \lim_{x\to 2} \frac{2-2}{2^3-8} = \frac{0}{0}$$ indeterminate form

factor

$$= \lim_{x\to 2} \frac{(x-2)}{(x-2)(x^2+2x+4)}$$

$$= \lim_{x\to 2} \frac{1}{x^2+2x+4} = \frac{1}{2^2+2(2)+4} = \boxed{\frac{1}{12}}$$

$$3.\ \lim_{x\to 5} \frac{x^2-25}{x-5}$$

$$= \frac{5^2-25}{5-5} = \frac{0}{0}$$

Factor

$$= \lim_{x\to 5} \frac{(x-5(x+5)}{(x-5)}$$

$$= \lim_{x\to 5} (x+5) = 5+5 = \boxed{10}$$

Limits involving $x \to \infty$

4 $\lim_{x\to\infty} \frac{x^2 - 6x^4 - 3}{x^4 + 2x - 1}$ using the short cut method

$= \frac{-6}{1} = \boxed{6}$

5 $\lim_{x\to\infty} \frac{x^{\frac{1}{3}}}{x^9+2} = \boxed{0}$

6. $\lim_{x\to\infty} \frac{x+2}{e^x}$ Use L'Hopital's Rule

$= \frac{\infty + 2}{e^{\infty}} = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = \boxed{0}$

7. $\lim_{x\to 0} \frac{\sin x}{2 - 2\cos x}$

$= \frac{\sin 0}{2 - 2\cos(0)} = \frac{0}{2-2} = \frac{0}{0}$ L'Hopitals

$= \lim_{x\to 0} \frac{\cos v}{2\sin x} = \frac{\cos(0)}{2\sin(0)} = \frac{1}{0}$ = undefined → $\boxed{\text{DNE}}$

8. $\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{e^{\infty}}{\infty^2} = \frac{\infty}{\infty}$ apply L'hopital's rule

$\lim_{x\to\infty} \frac{e'}{\ } = \frac{\infty}{\infty}$ Apply L'hopital's rule again

$\lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \boxed{\infty}$

Find the derivatives below:

1. $y = e^{2x} + \ln(x-4) + \sec x$

$y' = 2e^{2x} + \frac{1}{(x-4)} + \sec x \tan x$

2. $y = e^{-x} \sin e^{-x}$ (product rules)

$y' = -e^{-x} \sin e^{-x} + e^{-x} \cdot -e^{-x} \cos e^{-x}$

$= -e^{-x} \sin e^{-x} - e^{-2x} \cos e^{-x}$

3. $y = \ln(4x^2 - 3x - 1)$

$y' = \frac{8x - 3}{4x^2 - 3x - 1}$

4. $y = \frac{3e^x}{x^{-4}}$

$y' = \frac{3e^x(x^{-4}) - 3e^x(-4x^{-5})}{x^{-8}}$

$= \frac{3e^x(x^{-4} + 4x^{-5})}{x^{-8}}$

$= \frac{3e^x x^{-5}(x+4)}{x^{-8}}$

$= 3e^x x^3(x+4)$

or

$y = \frac{3e^x}{x^{-4}}$

$y = 3e^x \cdot x^4$

$y' = 3e^x x^4 + 3e^x \cdot 4x^3$

$y' = 3e^x x^3(x+4)$

5. $y = \frac{1}{\sqrt{3x-4}}$

$y = \frac{1}{(3x-4)^{1/2}}$

$y = (3x-4)^{-\frac{1}{2}}$

$y' = -\frac{1}{2}(3x-4)^{-3/2}(3)$

$y' = \frac{3}{2}\,(3x-4)^{-3/2}$

$y' = \frac{3}{2\,(3x-4)^{3/2}}$

$= \frac{3}{2\sqrt{(3x-4)^3}}$

$= \frac{3}{2(3x-4)\sqrt{3x-4}}$

6. $y = \cos^5 4x$

$y = (\cos 4x)^5$

$y' = 5(\cos 4x)^4 (-4 \sin 4x)$

$= -20(\sin 4x)(\cos^4 4x)$

7. $y = \sqrt{e^x}$

$y = (e^x)^{\frac{1}{2}}$

$y = e^{\frac{1}{2}x}$

$y' = \frac{1}{2} e^{\frac{1}{2}x}$

8. $y = \sin 3x^2$

$y' = 6x \cos 3x^2$

9. $y = x e^{x^2}$

$y' = 1\, e^{x^2} + x \cdot 2x\, e^{x^2}$

$= e^{x^2} + 2x^2 e^{x^2}$

$y' = e^{x^2}(1 + 2x^2)$

10. $y = \ln(\cos x)$

$y' = \frac{-\sin x}{\cos x}$

$y' = -\tan x$

11 $y = v^4 - 6x + 2$

$y' = 4x^3 - 6$

12. $y = \sqrt{\pi^{5e}}$ π and e are constants

$y' = 0$

13. $y = \sin 2x \sqrt{x^2+3}$

$y = \sin 2x \, (x^2+3)^{\frac{1}{2}}$

$y' = 2\cos 2x \, (x^2+3)^{\frac{1}{2}} + \sin 2x \cdot \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$

$= 2\cos 2x \sqrt{x^2+3} + x \sin 2x \, (x^2+3)^{-\frac{1}{2}}$

we can factor

$= (x^2+3)^{-\frac{1}{2}} \left[2\cos 2x \, (x^2+3) + x(\sin 2x)\right]$

$= \dfrac{2(x^2+3)\cos 2x + x \sin 2x}{\sqrt{x^2+3}}$

14 $y = (3x^3+4x-5)^{-3}$

$y' = -3(3x^3+4x-5)^{-4} \, (9x^2+4)$

$= \dfrac{-3(9x^2+4)}{(3x^3+4x-5)^4}$

Integrate

1. $\int e^{(3x+2)} dx$ $\quad u = 3x+2$, $du = 3\,dx$

$\frac{du}{3} = dx$

$= \int e^{u} \cdot \frac{du}{3}$

$= \frac{1}{3} \int e^{u} du$

$= \frac{1}{3} e^{u} + C$

$= \frac{1}{3} e^{(3x+2)} + C$

2. $\int x^2 e^{x^3} dx$ $\quad u = x^3$, $du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

$= \int x^2 e^{u} \cdot \frac{du}{3x^2}$

$= \frac{1}{3} \int e^{u} du$

$= \frac{1}{3} e^{u} + C$

$= \frac{1}{3} e^{x^3} + C$

3 $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$ $\quad u = \cos x$ $\quad du = -\sin x \, dx$

$\frac{du}{-\sin x} = dx$

$\int \frac{\sin x}{u} \cdot \frac{du}{\sin x}$

$= -\int \frac{du}{u} = \ln|\cos x| + C$

4. $\int \frac{1}{2x+7}\,dx$ $\qquad u = 2x+7 \qquad du = 2dx$

$\frac{du}{2} = dx$

$\int \frac{1}{u} \cdot \frac{du}{2}$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2} \ln|u| + C$

$\boxed{= \frac{1}{2} \ln|2x+7| + C}$

5. $\int_1^4 y^{-2}\,dx$

$= \frac{x^{-1}}{-1}\Big|_1^4$

$= -\frac{1}{x}\Big|_1^4 = \left(-\frac{1}{4}\right) - \left(-\frac{1}{1}\right)$

$= -\frac{1}{4} + 1$

$= \boxed{\frac{3}{4}}$

6 $\int \frac{\ln x}{x}\,dx$ $\qquad u = \ln x \qquad du = \frac{1}{x}dx$

$x\,du = dx$

$= \int \frac{\text{[illegible]}}{x} \cdot x\,du$

$= \int u\,du$

$= \frac{(u)^2}{2} + C = \boxed{\frac{(\ln x)}{2} + C}$

7. $\int \frac{\sin x}{1+2\cos x}\,dx$ $\qquad u = 1+2\cos x, \quad du = -2\sin x\,dx$

$\qquad\qquad\qquad\qquad \frac{du}{-2\sin x} = dx$

$= \int \frac{\sin x}{u} \cdot \frac{du}{-2\sin x}$

$= \frac{-1}{2}\int \frac{du}{u}$

$= -\frac{1}{2}\ln|u| + C$

$= \boxed{-\frac{1}{2}\ln|1+2\cos x| + C}$

8 $\int \frac{2x+1}{\sqrt{x^2+x}}\,dx$ $\qquad u = x^2+x \quad du = (2x+1)\,dx$

$\qquad\qquad\qquad\qquad \frac{du}{(2x+1)} = dx$

$\int \frac{(2x+1)}{(x^2+x)^{\frac{1}{2}}}\,dx$

$\int \frac{(2x+1)}{u^{\frac{1}{2}}} \cdot \frac{du}{(2x+1)}$

$= \int \frac{du}{u^{\frac{1}{2}}}$

$= \int u^{-\frac{1}{2}}\,du$

$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \boxed{2(x^2+x)^{\frac{1}{2}} + C}$

9 $\int \frac{x-1}{(x^2-2x+5)^3}\,dx$

$\boxed{u = x^2-2x+5}$

$du = (2x-2)\,dx$

$du = 2(x-1)\,dx$

$\boxed{\frac{du}{2(x-1)} = dx}$

$= \int \frac{(x-1)\cdot du}{u^3(2)(x-1)}$

$= \int \frac{du}{u^3}$

$= \int u^{-3}\,du$

$= \frac{u^{-2}}{-2} + c$

$= \frac{-1}{2u} + C$

$= \boxed{\frac{-1}{2(x^2-2x+5)}}$

10 $\int x\, e^{6x}\,dx$

Use by parts $u = x$ $dv = \int e^{6x}dx$

$du = dx$ $v = \frac{1}{6}e^{6x}$

$= uv - \int v\,du$

$= x\cdot\frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x}dx$

$= \frac{xe^{6x}}{6} - \frac{1}{6}\int e^{6x}\,dx$

$= \frac{xe^{6x}}{6} - \frac{1}{6}\cdot\frac{1}{6}e^{6x} + C$

$= \frac{xe^{6x}}{6} - \frac{1}{36}e^{6x} + c$ or $\frac{e^{6x}}{6}\left(x - \frac{1}{6}\right) + C$

11. $\int \frac{x+25}{x^2+2x-63}\,dx$ use partial fraction decomposition

$$= \int \frac{x+25}{(x-7)(x+9)}\,dx = \int \frac{A}{(x-7)}\,dx + \int \frac{B\,dx}{(x+9)}$$

We will solve for A and B in the expression below

$$\frac{x+25}{(x-7)(x+9)} = \frac{A}{(x-7)} + \frac{B}{(x+9)}$$

$$x+25 = A(x+9) + B(x-7)$$

$$x+25 = Ax + 9A + Bx - 7B$$

Let:

(1) $x = Ax + Bx$ (2) $25 = 9A - 7B$

(eliminate the x by dividing by x)

① $\frac{x}{x} = A\frac{x}{x} + \frac{Bx}{x}$

① $1 = A + B$ Use system of equation to solve.

① $1 = A + B$
② $25 = 9A - 7B$ $>$

$$\begin{aligned} 7 &= 7A + 7B \\ 25 &= 9A - 7B \\ \hline 32 &= 16A \\ 2 &= A \end{aligned}$$

plug this value in

(1) $1 = A + B$
$1 = 2 + B$
$\boxed{-1 = B}$

Back to the original integral

$$\int \frac{x+25}{(x-7)(x+9)}\,dx = \int \frac{2}{x-7}\,dx + \int \frac{-1}{x+9}\,dx$$

$$= 2\ln|x-7| - 1\ln|x+9| + c$$

$$= \ln \frac{(x-7)^2}{|x+9|} + C$$

12. $\int x \cos x \, dx$ by parts: $u = x$ $dv = \int \cos x \, dx$

$= uv - \int v \, du$ $du = dx$ $v = \sin x$

$= x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + C$

13. $\int \frac{x+18}{(x+5)(x-8)} dx = \int \frac{A}{(x\ 5} dx + \int \frac{B}{(x-8)} dx$

$\frac{x+18}{(x+5)(x-8)} = \frac{A}{(x+5)} + \frac{B}{(x-8)}$

$x+18 = A(x-8) + B(x+5)$

$x+18 = Ax - 8A + Bx + 5B$

$x = Ax + Bx$ ② $18 = -8A + 5B$

① $1 = A + B$

① $1 = A + B$ ② $18 = -8A + 5B$ $\Big\} = 8 = 8A + 8B$

$18 = -8A + 5B$

$26 = 13B$

$2 = B$

14. $\int 3 \ln x^5 \, dx$ by parts $u = \ln x^5$ $dv = \int dx$

$du = \frac{5x^4}{x^5} dx$ $v = x$

$= \frac{5}{x} dx$

$$3 \int \ln x^5 \, dx$$

$$= uv - \int v \, du$$

$$= 3\left[\ln x^5 (x) - \int x \cdot \frac{5}{x} dx\right]$$

$$= 3\left[x \ln x^5 - 5 \int dx\right]$$

$$= 3\left[x \ln x^5 - 5x\right] + C$$

$$= 3x\left[\ln x^5 - 5\right] + C$$

15. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ $\boxed{u = e^x + e^{-x}}$

$du = (e^x - e^{-x}) dx$

$\boxed{\frac{du}{(e^x - e^{-x})} = dx}$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{x}} dx$$

$$= \int \frac{(e^x - e^{-x})}{u} \cdot \frac{du}{(e^x - e^{-x})}$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|e^x + e^{-x}| + C$$

Applications

1. Find the equation of the tangent line

a. $y = (4x^2 - 8x + 3)^4$ at $P(2,8)$

$$y' = 4(4x^2 - 8x + 4)^3 (8x - 8)$$

at $P(2,8)$

$$m_{tan} = 4[4(2)^2 - 8(2) + 4]^3 [8(2) - 8]$$
$$= 4[16 - 16 + 4]^3 [16 - 8]$$
$$= 4[4]^3 (8)$$
$$= 4(64)(8)$$

Slope $= 2048$

Eq of the tangent line

$$y - y_1 = m(x - x_1)$$
$$y - 8 = 2048(x - 2)$$
$$y - 8 = 2048x - 4096$$
$$y = 2048x - 4088$$

2 Find the equation of the tangent line at P(2, 1) given:

$$x^2+y^2-2x+3y=5$$

Implicit differentiation

$$2x\frac{dx}{dx}+2y\frac{dy}{dx}-2\frac{dx}{dx}+3\frac{dy}{dx}=0$$

$$2x+2y\frac{dy}{dx}-2+3\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}+3\frac{dy}{dx}=2-2x$$

$$(2y+3)\frac{dy}{dx}=2-3x$$

$$\frac{dy}{dx}=\frac{2-2x}{2y+2}$$

at P(2, 1), plug in the values for x and y.

$$\frac{dy}{dx}=\frac{2-2(2)}{2(1)+3}=\frac{-2}{5}$$

$$\boxed{\text{Slope}=-\frac{2}{5}}$$

Eq of the tangent line

$$y-y_1=m(x-x_1)$$

$$y-1=-\frac{2}{5}(x-2)$$

$$y-1=-\frac{2}{5}x+\frac{4}{5}$$

$$y=-\frac{2}{5}x+\frac{4}{5}+1$$

$$\boxed{y=-\frac{2}{5}x+\frac{9}{5}}$$

3. Find $\frac{dy}{dt}$: given $xy = 9$ subject to
$\frac{dx}{dt} = 2$ when $x = 3$

$$xy = 9$$

Note: xy is a product

$$1\frac{dx}{dt}\cdot y + x \cdot 1\frac{dy}{dt} = 0$$

$$y\frac{dx}{dt} + x\frac{dy}{dt} = 0$$

$$\frac{x\,dy}{dt} = -y\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{y\cdot\frac{dx}{dt}}{x}$$

$$\frac{dy}{dt} = -\frac{3(2)}{3}$$

$$= -2$$

Note: $xy = 9$ if
$x = 3$ then $xy = 9$
$3y = 9$
$y = 3$

4. Find y''' given $y = x^4 + \sin x$

$$y = x^4 + \sin x$$

$$y' = 4x^3 + \cos x$$

$$y'' = 12x^2 - \sin x$$

$$y''' = 24x - \cos x$$

5. Solve the differential equation subject to the given conditions

$\frac{d^2y}{dx^2} = 6x - 4 \qquad y = 4 \quad y' = 5$ if $x = 2$

$\int \frac{d^2y}{dx^2}\,dx = \int (6x-4)\,dx$

(1) $\frac{dy}{dx} = 3x^2 - 4x + C$ We need to solve for C

use the conditions $x = 2$ and $y' = 5$

$$5 = 3(2)^2 - 4(2) + C$$
$$5 = 12 - 8 + C$$
$$5 - 4 = C$$
$$1 = C$$

going back to (1) above

$$\frac{dy}{dx} = 3x^2 - 4x + C$$
$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Integrating again

$$\int \frac{dy}{dx} = \int (3x^2 - 4x + 1)\,dx$$

$y = x^3 - 2x^2 + x + C$ use the condition $y = 4$ and $x = 2$

$$4 = 2^3 - 2(2)^2 + 2 + C$$
$$4 = 8 - 8 + 2 + C$$
$$2 = C$$

$$\boxed{\therefore\ y = x^3 - 2x^2 + x + 2}$$

6. Let $v(t)$ represent the velocity at t find $s(t)$ and $a(t)$, where $s(t)$ is the position at time t and $a(t)$ represents the acceleration.

Given: $v(t) = 3t^2 + 4t^4 + e^{3t}$

$$s(t) = \int v(t)$$

$$\int v(t) = \int (3t^2 + 4t^4 + e^{3t})\,dt$$

$$v(t) = t^3 + \frac{4}{5}t^5 + \frac{1}{3}e^{3t} + C$$

$$a(t) = v'(t)$$

$$a(t) = 6t + 16t^3 + 3e^{3t}$$

7. Determinant

a. Evaluate the determina~~tts~~ below:

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 4 \\ -1 & -1 & -5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 4 \\ -1 & -5 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 4 \\ -1 & -5 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 1(-10+4) + 1(-10+4) + 3(-2+2)$$

$$= 1(-6) + 1(-6) + 3(0)$$

$$= -6 - 6$$

$$\boxed{= 12}$$

b. Find the cofact for component a_{21}

$$-2\begin{vmatrix} -1 & 3 \\ -1 & -5 \end{vmatrix}$$

8. Find the equation of a line passing through P(3,-3) and parallel to the line $y = 7x+3$

We have a point P given. We need a slope for the new line. We know that if lines are parallel their slopes are the same. So the slope of our new line must the the same as the slope of the given line, which is 7.

$$\therefore \quad y+3 = 7(x-3)$$

$$y = 7x-21-3$$

$$y = 7x-24$$

9. Solve the system of equations

(1) $y^2 = x$

(2) $x+2y-3=0$

Use the substitution method. Substitute $x=y^2$ in eq (2)

$$x+2y-3=0$$

$$\downarrow$$

$$y^2+2y-3=0$$

$$(y+3)(y-1)=0$$

$$y=-3, 1$$

Now we need to find x-values for each of the y using equation (1)

$y^2 = x$

$(-3)^2 = x$

$9 = x$

$y^2 = x$

$(1)^2 = x$

$1 = x$

$\boxed{(1,1) \text{ and } (9,-3) \quad \text{Solution}}$

Notes

10 If $\int_1^2 f(x)\,dx = -4$ and $\int_1^5 f(x)\,dx = 6$

find $\int_2^5 f(x)\,dx$ given:

$$\int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx$$

from the given information we get

$$-4 + \int_2^5 f(x)\,dx = -6$$

$$\int_2^5 f(x)\,dy = -6 + 4$$

$$= \boxed{-2}$$

PHYSICS

REVIEW

Chapter 3

Motion in One Dimension

The study of motion is called **kinematics**, and it is here that we begin our study of physics. We will follow a method that has proven very effective in science. We start by studying simple situations and then study gradually more complex physical problems. In this chapter we consider motion in one dimension, without regard to the forces that influence the motion. In the next chapter we will extend the discussion to motion in two or three dimensions, but first we need a good understanding of the basic concepts involved – namely, displacement, velocity, and acceleration.

3.1 DISPLACEMENT AND VELOCITY

The position of an object moving along the x axis is described by its x coordinate. The change in the object's position is its displacement Δx. If the object is at position x_1 at time t_1 and at x_2 at time t_2, then $\Delta x = x_2 - x_1$. Displacement is a vector. However, for motion in one dimension we can specify the displacement simply in terms of the x coordinate of the particle. If the particle is to the right of the origin, its coordinate is positive. If it is to the left of the origin, its coordinate is negative. We define the **average velocity** v as

$$\bar{v} \equiv \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \tag{3.1}$$

If we choose our origin such that $x_1 = 0$ and $t_1 = 0$, then the position x at later time t is $x = vt$.

3.2 INSTANTANEOUS VELOCITY AND ACCELERATION

If an object experiences a displacement Δx in a time Δt, its **instantaneous velocity** is

$$\boxed{v \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}} \tag{3.2}$$

Velocity is a vector, but in one dimension we can indicate direction merely by giving the sign of the velocity. The magnitude of velocity is called **speed**. Speed is what a car's speedometer measures. Speed is always positive. Speed and velocity are measured in meters per second. **Velocity is the slope of a graph of *x* versus *t***, as illustrated in Figure 3.1. When the slope is positive, the object is moving to the right. When the slope is negative, the object is moving left. When the slope is zero, the object is stopped.

The rate at which velocity is changing is measured by **acceleration.** Thus if an object has velocity v_1 at time t_1 and velocity v_2 at time t_2, its **average acceleration** is

$$\bar{a} \equiv \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \tag{3.3}$$

and its **instantaneous acceleration** is

$$\boxed{a \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}} \tag{3.4}$$

Acceleration has units of velocity/time: m/s/s = m/s^2.

When thinking about the units of acceleration, never say to yourself, "meters per square second." Instead, always say, "meters per second (pause) per second." This makes clear the idea that acceleration is a measure of how much the velocity is changing each second. Drag racers describe a car's acceleration in units of "miles per hour per second." Thus if a car can go from zero to 60 mi/h in 6 s, its acceleration is 10 mi/h · s. We always measure acceleration in units of meters per second per second, but the drag racer's mixed units convey the idea of acceleration more clearly.

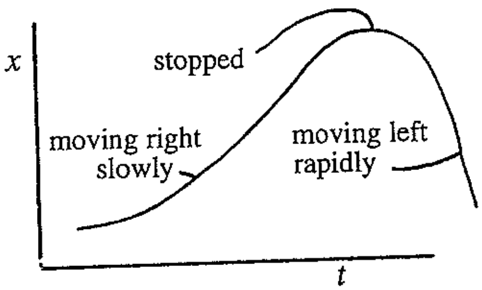


Figure 3.1

It is best to avoid use of the common word "deceleration." Describe acceleration simply as positive or negative. Note that *negative acceleration does not necessarily mean "slowing down."* When velocity and acceleration both have the same sign, the object speeds up. When velocity and acceleration have opposite signs, the object slows down.

Illustrative graphs of displacement, velocity, and acceleration for a moving object are shown in Figure 3.2. Note that v can be deduced from the x versus t curve by remembering that v is the slope of x versus t. Similarly, a can be deduced from v versus t, since a is the slope of v versus t.

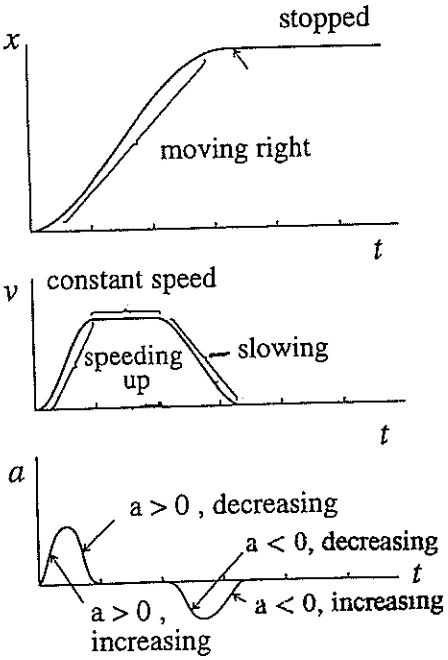


Figure 3.2

Acceleration is the **second derivative of displacement.** Thus

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{3.5}$$

3.3 CONSTANT ACCELERATION

Many interesting phenomena involve motion with constant acceleration. In this case it is easy to obtain expressions for velocity and displacement by integrating the acceleration. Thus if

$$a = \frac{dv}{dt} = \text{constant}$$

then

$$v = \int a\,dt = at + c_1$$

We can determine the constant by observing that if at time $t = 0$ the velocity has initial value v_0, then $v_0 = 0 + c_1$, so $c_1 = v_0$ and

$$v = at + v_0 \tag{3.6}$$

We can integrate the velocity to obtain the displacement x.

$$v = \frac{dx}{dt}$$

so

$$x = \int v\,dt = \int (at + v_0)\,dt$$

$$= \frac{1}{2}\,at^2 + v_0 t + c_2$$

If at time $t = 0$, the value of x is x_0 (the initial position), then $x_0 = c_2$ and

$$x = x_0 + v_0 t + \frac{1}{2}\,at^2 \tag{3.7}$$

In most problems it is convenient to choose the origin at the position of the object at $t = 0$, that is, to set $x_0 = 0$. When this is done, Eq. 3.7 becomes

$$x = v_0 t + \tfrac{1}{2}at^2 \tag{3.8}$$

We can check that Eqs. 3.6 and 3.8 are correct by differentiating them. Thus the derivative of x yields the correct expression for v, and the derivative of v yields the constant acceleration a.

We can solve Eq. 3.6 for t and substitute the result in Eq. 3.8. When this is done, we obtain

$$t = \frac{v - v_0}{a} \qquad x = v_0\frac{(v - v_0)}{a} + \frac{1}{2}\,a\frac{(v - v_0)^2}{a^2}$$

$$2ax = 2v_0v - 2v_0^2 + v^2 - 2v_0v + v_0^2$$

$$v^2 = v_0^2 + 2ax \tag{3.9}$$

The above equations are so important that it is worthwhile to place them all together and memorize them.

$$\boxed{\begin{aligned} &\text{If } a = constant, \text{ then} \\ &\quad v = v_0 + at \\ &\quad x = x_0 + v_0t + \tfrac{1}{2}at^2 \\ &\quad v^2 = v_0^2 + 2ax \end{aligned}} \tag{3.10}$$

The case of zero acceleration (constant velocity) is important and results in simple equations.

$$\boxed{\begin{aligned} &\text{Thus if } a = 0, \text{ then} \\ &\qquad v = v_0 \text{ (constant)} \\ &\qquad x = v_0t \end{aligned}} \tag{3.11}$$

CAUTION: Do not use Eq. 3.11 if acceleration is not zero. Failure to heed this admonition is a common source of error.

Problem 3.1 A motorist drives for 2 h at 100 km/h and for 2 h at 80 km/h. What is her average speed?

Solution $\bar{v} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{(100\text{km/h})\,(2\text{ h}) + (80\text{ km/h})(2\text{ h})}{2\text{ h} + 2\text{ h}} = 90\text{ km/h}$

Problem 3.2 A motorist drives 120 km at 100 km/h and 120 km at 80 km/h. What is his average speed for the trip?

Solution $\bar{v} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{120\text{ km} + 120\text{ km}}{120\text{km}/100\text{km/h} + 120\text{km}/80\text{km/h}} = \dfrac{240\text{ km}}{1.2\text{ h} + 1.5\text{ h}} = 88.9\text{ km/h}$

Observe that in Problem 3.1, the average speed was halfway between the high speed and the low speed, because the motorist drove equal *times* at each speed. Here, however, the motorist drove equal *distances* at each speed but drove for a longer time at the lower speed, so the average speed is closer to the lower speed and is *not* halfway in between. Remember, *average* means "time average."

Problem 3.3 In good weather the drive from Seattle to Spokane, Washington, on Interstate 90 takes 3 h 51 min at an average speed of 105 km/h. In winter, however, it is not unusual to average only 80 km/h. How long would the trip take at this average speed?

Solution Express the time in hours. Thus $t_1 = 3\ \text{h}\,51\,\text{min} = (3 + 51/60)\ \text{h} = 3.85\ \text{h}$. Since $x = v_1 t_1 = v_2 t_2$, then

$$t_2 = \frac{v_1}{v_2} t_1 = 105/80\ (3.85\ \text{h}) = 5.05\ \text{h}$$

$$= 5\ \text{h} + (0.05\ \text{h})(60\ \text{min/h}) = 5\ \text{h}\,3\ \text{min}$$

Problem 3.4 A cheetah is the fastest land mammal, and it can run at speeds of about 101 km/h for a period of perhaps 20 s. The next fastest land animal is an antelope, which can run at about 88 km/h for a much longer time. Suppose a cheetah is chasing an antelope, and both are running at top speed. (a) If the antelope has a 40-m head start, how long will it take the cheetah to catch him, and how far will the cheetah travel in this time? (b) What is the maximum head start the antelope can have if the cheetah is to catch him within 20 s (at which time the cheetah runs out of breath)?

Solution (a) The speeds are constant, so Eq. 3.10, $x = vt$, applies. Both animals run for the same time, but the cheetah must run 40 m extra. Thus

$$x_C = v_C t = x_A + 40 \qquad (i)$$

and

$$x_A = v_A t \qquad (ii)$$

Substitute *ii* in *i* and solve for *t*:

$$v_C t = v_A t + 40 \qquad (v_C - v_A)t = 40 \qquad t = \frac{40}{v_C - v_A}$$

The speeds must be expressed in meters per second, not kilometers per hour.

$$v_C = 101\ \text{km/h} = 101\left(\frac{1609\ \text{m}}{3600\ \text{s}}\right) = 45.1\ \text{m/s}$$

$$v_A = 88\ \text{km/h} = 39.3\ \text{m/s} \qquad t = \frac{40\ \text{m}}{(45.1 - 39.3)\text{m/s}} = 6.9\ \text{s}$$

(b) Let $h =$ head start distance and $t = 20$ s for both animals. If the cheetah is to catch the antelope, then $x_C = x_A + h$.

$$x_C = v_C t \qquad x_A = v_A t$$

So $v_Ct = v_At + h \qquad h = (v_C - v_A)t = (45.1 - 39.3)(20) = 116\,\mathrm{m}$

Problem 3.5 A typical jet fighter plane launched from an aircraft carrier reaches a take-off speed of 175 mi/h in a launch distance of 310 ft. (a) Assuming constant acceleration, calculate the acceleration in meters per second per second. (b) How long does it take to launch the fighter?

Solution (a) The plane starts from rest, so $v_0 = 0$. From Eq. 3.10 I choose the equation relating v, v_0, a, and x. I use this equation because I know v, v_0, and x and I want to find a. I do not use an equation involving t since I do not yet know t. If you were to start with one of the other equations, you would eventually reach the correct answer, but more algebra would be involved. With practice you will learn which equation to use for the easiest solution. Using Table 2.1 convert the data to SI units:

$$v = 175\ \mathrm{mi/h} = (175)(0.447\ \mathrm{m/s}) = 78.2\ \mathrm{m/s}$$

$$x = 310\ \mathrm{ft} = (310)(0.305\ \mathrm{m}) = 94.6\ \mathrm{m}$$

$$v^2 = v_0^2 + 2\,ax = 0 + 2\,ax$$

$$a = \frac{v^2}{2x}$$

$$= \frac{(78.2\ \mathrm{m/s})^2}{2(94.6\ \mathrm{m})} = 32.3\ \mathrm{m/s^2}$$

(b) From Eq. 3.10, $v = v_0 + at = 0 + at$, so

$$t = \frac{v}{a} = \frac{78.2\ \mathrm{m/s}}{32.3\ \mathrm{m/s^2}} = 2.4\ \mathrm{s}$$

Problem 3.6 A motorist traveling 31 m/s (about 70 mi/h) passes a stationary motorcycle police officer. 2.5 s after the motorist passes, the police officer starts to move and accelerates in pursuit of the speeding motorist. The motorcycle has constant acceleration of 3.6 m/s^2. (a) How fast will the police officer be traveling when he overtakes the car? Draw curves of x versus t for both the motorcycle and the car, taking $t = 0$ at the moment the car passes the stationary police officer. (b) Suppose that for reasons of safety the policeman does not exceed a maximum speed of 45 m/s (about 100 mi/h). How long will it then take him to overtake the car, and how far will he have traveled?

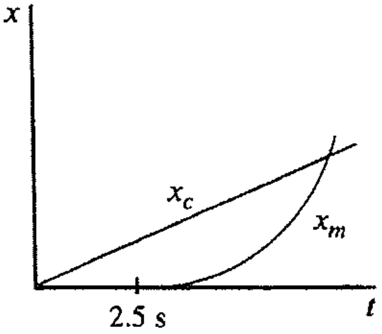
Solution (a) The car has constant velocity and travels a distance x_c in time t:

$$x_c = v_ct$$

The motorcycle starts from rest ($v_0 = 0$) and moves a distance x_m in time $t - 2.5$ with constant acceleration:

$$x_m = \frac{1}{2}\,a(t-2.5)^2$$

These curves are sketched here. When the motorcycle overtakes the car, both will have traveled the same distance. Thus



$$\frac{1}{2}\,a(t-2.5)^2 = v_c t$$

Substitute numerical values, and solve this quadratic equation for t, using Eqs. 1.31 and 1.32.

$$\frac{1}{2}\,(3.6)(t-2.5)^2 = 31t \qquad 1.8t^2 - 9t + 11.25 = 31t$$

$$1.8t^2 - 42t + 11.25 = 0$$

$$t = \frac{42 \pm \sqrt{(42)^2 - 4(1.8)(11.25)}}{(2)(1.8)} \qquad t = 0.27\,\text{s or } 23\,\text{s}$$

The motorcycle did not start until $t = 2.5$ s, so the solution we want is $t = 23$ s.

$$v_m = v_0 + at = 0 + (3.6\,\text{m/s}^2)(23\,\text{s}) = 83\,\text{m/s} = 186\,\text{mi/h}$$

(b) Suppose the motorcycle accelerates for time t_1 over distance x_1 to a maximum speed $v = 45$ m/s. It then continues at constant speed v for time t_2 and distance x_2 until it catches the car. The variables are then related as follows:

$$x_c = v_c(t_1 + t_2 + 2.5)\ \ (i) \qquad x_1 = \tfrac{1}{2}at_1^2\ \ (ii) \qquad x_2 = v_m t_2\ \ (iii)$$

$$x_c = x_1 + x_2\ \ (iv) \qquad v_m = at_1\ \ (v)$$

The preceding are five equations in five unknowns: x_1, x_2, t_1, t_2, and x_c. They can be solved simultaneously. The values of v_m and a are known, so Eq. *v* gives t_1 immediately. Substitute this value for t_1 in Eq. *ii* and x_1 is obtained. Now use Eqs. *i*, *iii*, and *iv* to solve for the remaining three variables, x_2, t_2, and x_c. The results are $t_1 = 12.5$ s, $x_1 = 281$ m, $x_2 = 591$ m, $t_2 = 13.1$ s, $x_c = 872$ m, and $t = t_1 + t_2 = 25.6$ s.

Problem 3.7 Suppose that motion studies of a runner show that the maximum speed he can maintain for a period of about 10 s is 12 m/s. If in a 100-m dash this runner accelerates with constant acceleration until he reaches this maximum speed and then maintains this speed for the rest of the race, what acceleration will he require if his total time is 11 s?

Solution Break the problem into two parts. While accelerating, the runner travels a distance x_1 in time t_1, and then he runs the remaining distance x_2 in time t_2 at constant speed v.

$$x_1 + x_2 = 100 \quad (i) \qquad t_1 + t_2 = 11 \quad (ii)$$

$$x_1 = \tfrac{1}{2}\, at_1^2 \quad (iii) \qquad v = at_1 \quad (iv) \qquad x_2 = vt_2 \quad (v)$$

These five equations can be solved for the five unknowns: a, x_1, x_2, t_1, and t_2. Substitute (*iii*) and (*v*) into (*i*): $1/2\ at_1^2 + vt_2 = 100$. Solve (*ii*) for t_2 and substitute into the above equation, first multiplying by 2:

$$at_1^2 + 2v(11 - t_1) = 200$$

Solve (*iv*) for t_1 and substitute it in the above equation:

$$a(\tfrac{v}{a})^2 + 2v(11 - \tfrac{v}{a}) = 200$$

Solve for a:

$$a = \frac{v^2}{22\,v - 200}$$

Substitute $v = 12$ m/s. The result is $a = 2.25$ m/s^2.

In solving complicated problems like this, first be sure you are clear about exactly what is happening. Draw a little picture with a runner and the different parts of the race indicated. Label all the relevant quantities, using different symbols for the different unknown quantities. Here we had five unknowns, so we know we will need five independent equations to solve the problem. Write down the equations, and then solve them using algebra. Finally, substitute numerical values.

3.4 FREELY FALLING BODIES

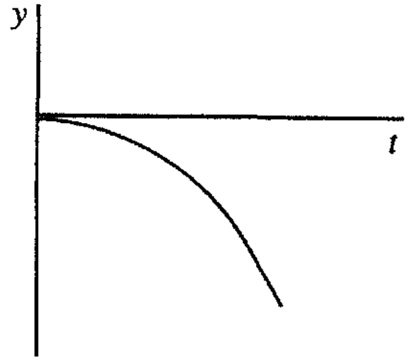
Consider an object moving upward or downward along a vertical axis. Let us neglect any air effects and consider only the influence of gravity on such an object. It has been found that all objects, large and small, experience the same acceleration due to the force of gravity. This acceleration varies slightly with altitude, but for objects near the surface of the earth the acceleration is approximately constant. The acceleration is *always* directed downward, since it is caused by the downward force of gravity. We label the vertical axis the y axis, with upward taken as the positive direction. We take $y = 0$ at some convenient point, such as sea level or floor level. We call the magnitude of the acceleration due to the force of gravity g. The value of g is approximately 9.80 m/s^2. The value of g is slightly smaller high in the mountains and slightly larger at low elevations, such as in Death Valley. Note that the acceleration of an object acted on only by the force of gravity is $-g$, since the acceleration is downward and hence negative. This is true whether the object is falling downward, moving upward, or momentarily stopped at its highest point. Equation 3.10 describes this situation. Taking y as our independent variable and setting $a = -g =$ constant, these equations become

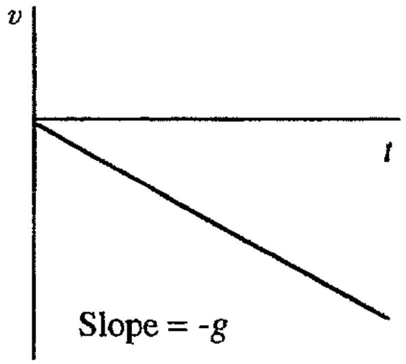
$$\boxed{\begin{aligned} a &= -g \\ v &= v_0 - gt \\ y &= y_0 + v_0 t - 1/2gt^2 \end{aligned}} \qquad (3.12)$$

In all of the following, neglect the influence of air. This is a fair approximation for objects that do not fall too far or too fast. Later we will see how to incorporate air drag.

Problem 3.8 A rock is dropped from rest from the Golden Gate Bridge. How far will it have fallen after 1 s? After 2 s? After 3 s? How fast will it be moving at each time?

Solution It is convenient to take the starting position of the rock as $y_0 = 0$. Thus subsequent y values will be negative as the rock falls. The rock is initially at rest, so $v_0 = 0$. Thus Eq. 3.12 yields $y = -1/2\ gt^2$ and $v = -gt$. Substituting $t = 1$, 2, and 3 s yields $y\ (1) = -4.9$ m, $v\ (1) = -9.8$ m/s, $y\ (2) = -19.6$ m, $v\ (2) = -19.6$ m/s, $y\ (3) = -4.1$ m, and $v\ (3) = -29.4$ m/s. Graphs of y versus t and v versus t are shown below. The graph of y is a parabola. Each succeeding second, the rock falls a greater distance as it gains speed. Note that the slope of the v versus t curve (the acceleration) is constant and negative.





Problem 3.9 Using a slingshot, a kid shoots a rock straight up at 30 m/s from the top of the Rogun Dam (the world's highest dam) in Tajikistan. It finally strikes the water 325 m below its starting point. (Assume the face of the dam is vertical. Actually the dam slopes outward, but let's neglect this slight complication for now.) How high does the rock rise? How long is it in the air? How long would it have been in the air if it had been launched straight down? How long if dropped from rest? Sketch a graph of v versus t for each of these three cases.

Solution Take $y = 0$ at the initial position. Let $y = h$ be the highest point reached. Let $y = -H$ be the surface of the water below. The initial velocity if thrown upward is $v_0 = 30\,\text{m/s}$. At the highest point, $v = 0$. Thus from Eq. 3.12,

$$v^2 = v_{-0}^2 - 2gh = 0$$

so
$$h = \frac{v_0^2}{2g} = \frac{30^2\,\text{m}}{2(9.8)} = 45.9\,\text{m}$$

At the water $y = -H$, so Eq. 3.12 yields $-H = 0 + v_0t - 1/2\ gt^2$ where t is the time in the air. Solve this quadratic equation using Eq. 1.31.

$$4.9t^2 - 30t - 325 = 0 \qquad t = \frac{30 \pm \sqrt{(-30)^2 - 4(4.9)(-325)}}{2(4.9)} \qquad t = -5.64\,\text{s or } 11.8\,\text{s}$$

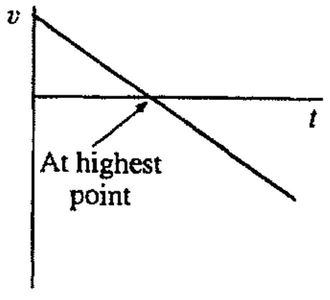
The physically meaningful solution is 11.8 s. The solution –5.64 *s* is allowed by the equations, but it does not correspond to what is happening here. If you imagined that the rock was projected upward from the water surface (where $y = -H$) at a time 5.64 s earlier than the actual launch, the equation is satisfied, but this is not what actually happened.

If the rocket was initially launched straight down, then $v_0 = -30\,\text{m/s}$ and the equation for y becomes

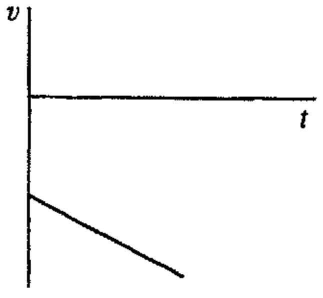
$$-H = 0 + v_0t - \tfrac{1}{2}gt^2 \qquad \text{or} \qquad -325 = -30t - 4.9t^2$$

Solving for t as before, $t = 5.64\,\text{s}$.

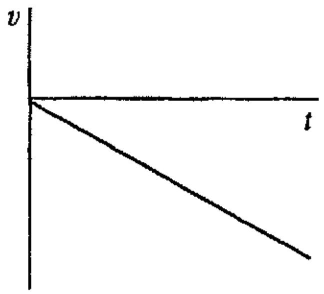
If dropped from rest, $v_0 = 0$, and $-H = 0 + 0 - 1/2\ gt^2$, so $t = 8.14$ s.



Thrown Upward



Thrown Downward



Dropped From Rest

Note that when $v = 0$ for the rock thrown upward, it is at its highest point. Here the velocity is momentarily zero; a fraction of a second later it is moving downward and has negative velocity.

Problem 3.10 A ball is thrown straight up. Show that it spends as much time rising as it does falling back to its starting point.

Solution At the peak of its flight $v = 0$. Thus $v = v_0 - gt_1 = 0$. Rise time is thus $t_1 = v_0/g$. Elevation is given by $y = v_0 t - 1/2\ gt^2$ assuming $y_0 = 0$.

When the ball returns to its starting point, $y = 0$. Thus $y = 0 = v_0 t - 1/2\ gt^2$, or $t = 2v_0/g = 2t_1$. The total time in the air is twice the rise time, so fall time = rise time.

3.5 SUMMARY OF KEY EQUATIONS

Average velocity: $\bar{v} = \frac{x_2 - x_1}{t_2 - t_1}$

Instantaneous velocity: $v = \frac{dx}{dt}$

Average acceleration: $\bar{a} = \frac{v_2 - v_1}{t_2 - t_1}$

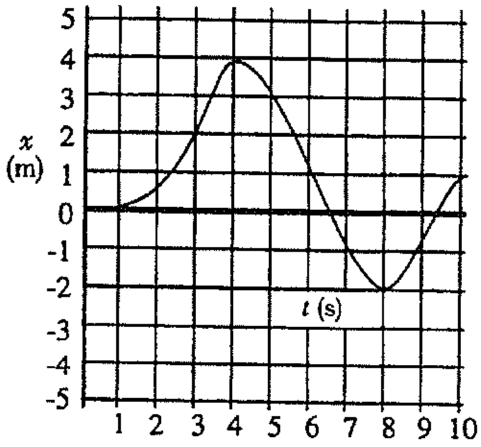
Instantaneous acceleration: $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

If $a =$ constant, then:	If $a = 0$, then:
$v = v_0 + at$	$v = v_0$ (constant)
$x = x_0 + vt + \frac{1}{2}\ at^2$	$x = vt$
$v^2 = v_0^2 + 2\,ax$	
For a freely falling object:	An object dropped from rest will fall a distance h in time t where:
$a = -g$	
$v = v_0 - gt$	$h = \frac{1}{2}\ gt^2$
$y = y_0 + v_0 t - \frac{1}{2}\ gt^2$	

Supplementary Problems

SP 3.1 A graph of the displacement of a moving particle as a function of time is shown here. For this time interval, determine

(a) How many times the particle stopped
(b) The total distance traveled
(c) When the particle was moving fastest
(d) How many times the particle returned to its starting point
(e) The direction the particle was moving at $t = 6$ s



SP 3.2 A Ferrari sports car can accelerate from rest to 96 km/h (about 60 mi/h) in 2.2 s. What is its average acceleration?

SP 3.3 On a 40-km bike ride a cyclist rides the first 20 km at 20 km/h. What speed is required for the final 20 km if the average speed for the trip is to be (a) 10 km/h? (b) 30 km/h? (c) If the cyclist rides very fast for the final 20 km, what is the maximum value his average speed could approach?

SP 3.4 The altitude of the space shuttle during the first 30 s of its ascent is described by the expression $y = bt^2$, where the constant b has the value 2.9 m/s². Using calculus, obtain an expression for the velocity and acceleration of the space shuttle during this period.

SP 3.5 When a girder in a bridge undergoes small vibrations, its displacement can be described by $x = A \cos 2\pi ft$, where A is the amplitude of the vibration (that is, the maximum value of x) and f is the frequency of the vibration. Such motion is called *simple harmonic motion*. Derive expressions for the velocity and acceleration for such motion.

SP 3.6 The leaning tower of Pisa is 54.5 m tall. Supposedly Galileo investigated the behavior of falling objects by dropping them from the top of this tower and timing their descent. How long would it take an object to fall 54.5 m if air effects are negligible?

SP 3.7 In 1939 a baseball nut named Joe Sprinz tried to set a world record for catching a baseball dropped from the greatest height. He tried to catch a ball dropped from a blimp 800 ft above him. On his fifth try he succeeded, but the ball slammed the glove into his face, breaking his jaw in 12 places, knocking out five teeth, and knocking him unconscious. As you might have guessed, the ball was really moving when it reached him. Air slows the ball appreciably, but you can get a pretty good idea of how fast the ball was moving even if you neglect air effects. How long would you estimate the ball took to fall 800 ft, and how fast was it going when it hit Joe?

SP 3.8 A car is moving 60 km/h when the driver sees a signal light 40 m ahead turn red. The car can slow with acceleration $-0.5g$ (where $g = 9.80$ m/s²). What is her stopping distance assuming (a) zero reaction time? (b) A reaction time of 0.20 s between when she sees the red light and when she hits the brake?

SP 3.9 A balloonist at an altitude of 800 m drops a package. One second later he drops a second package. (a) How far apart are the packages at the instant the second package is dropped? (b) How far apart are the packages when the first package hits the ground? (c) What is the time interval between when the two packages hit the ground?

SP 3.10 A girl on top of a building drops a baseball from rest at the same moment a boy below throws a golf ball upward toward her with a speed of 20 m/s. The golf ball is thrown from a point 18 m below where the baseball is released. How far will the baseball have dropped when it passes the golf ball?

SP 3.11 A typical jet liner lands at a speed of 100 m/s (about 224 mi/h). While braking, it has an acceleration of -5.2 m/s². (a) How long does it take to come to a stop? (b) What is the minimum length of the landing strip under these conditions?

SP 3.12 A movie stunt man wishes to drop from a freeway overpass and land on the roof of a speeding truck passing beneath him. The distance he will fall from rest to the roof of the truck is 12 m, and the truck is moving 80 km/h. What horizontal distance away should the truck be when the stunt man jumps?

SP 3.13 A ball is thrown upward with speed 12 m/s from the top of a building. How much later must a second ball be dropped from the same starting point if it is to hit the ground at the same time as the first ball? The initial position is 24 m above the ground.

SP 3.14 An object (perhaps a paratrooper) falls from an airplane and drops a vertical distance h. Upon striking snow-covered ground, the object stops with uniform acceleration a in a very small distance d. Determine the ratio a/g. A human has about a 50 percent chance of survival in such a fall if his or her acceleration does not exceed $50g$. Pilots have survived falls of 20,000 ft without a parachute, provided they landed in snow.

SP 3.15 The stopping distance of a car depends on its speed in a way that is counterintuitive for many people. The stopping distance is *not* simply proportional to the speed; that is, if you double your speed, you do not merely double your stopping distance. On a dry road a car with good tires may be able to obtain a braking acceleration of $-4.90\ \text{m/s}^2$. Calculate the stopping distance for a speed of 50 km/h and for 100 km/h.

SP 3.16 In a movie the FBI is investigating an assassination attempt on the life of the president. The setting is a parade in New York, and an amateur photographer has made a videotape of the passing motorcade. A careful examination of the tape shows in the background a falling object that turns out to be a pair of binoculars used by the would-be assassin. From the tape the FBI is able to determine that the binoculars fell the last 12 m before hitting the ground in 0.38 s. It is vital to them to know the height, and hence the building floor, from which the binoculars were dropped. Can this be determined from the given information? If so, from what height were the binoculars dropped?

SP 3.17 The earth travels in a nearly circular orbit about the sun. The mean distance of the earth from the sun is 1.5×10^{11} m. What is the approximate speed of the earth in its orbit around the sun?

SP 3.18 A driver traveling 100 km/h on a road in Montana sees a sheep in the road 32 m ahead. His reaction time is 0.70 s, and his braking acceleration is $-0.6g$ (where $g = 9.8\ \text{m/s}^2$). Is he able to stop before he hits the animal? If so, what is his stopping distance? If not, at what speed does he hit the sheep?

SP 3.19 Engineers at the Rand Corporation have designed a very high speed transit (VHST) vehicle that could radically reduce travel time between Los Angeles and New York. The 100-passenger car would be magnetically levitated and travel in an evacuated tube below the earth's surface. On the 4800-km (3000-mi) trip from LA to New York, the car would accelerate for the first half of the trip and then coast to a stop in New York. On both legs of the trip the acceleration would have constant magnitude. An acceleration of about $0.4g$ (where $g = 9.8\ \text{m/s}^2$) is the maximum a passenger can tolerate comfortably. Under these conditions, how long would the trip take? What maximum speed would be reached?

SP 3.20 In an accident on a freeway a sports car made skid marks 240 m long on the pavement. The police estimated the braking acceleration of the car to be $-0.9g$ under the road conditions prevailing. If this were true, what was the minimum speed of the sports car when the brakes were applied?

Solutions To Supplementary Problems

SP 3.1 (a) The particle is stopped when the slope is zero, that is, twice, at 4 s and 8 s.

(b) The particle first went 4 m to the right, then returned to its starting point and continued on 2 m to the left. It then went back to the right a distance of 3 m, for a total distance moved of $4 + 4 + 2 + 3 = 13\ m$.

(c) The particle is moving fastest when the slope is greatest, which is near $t = 3$ s.

(d) The particle returned twice to its starting point at $x = 0$.

(e) At $t = 6$ s the slope is negative, so the particle is moving to the left.

SP 3.2

$$v_0 = 0 \qquad v = 96\ \text{km/h} = 96(1000\ \text{m/3600 s}) = 26.7\ \text{m/s}$$

$$v = v_0 + at \qquad a = \frac{v - v_0}{t} = \frac{26.7 - 0}{2.2}\ \text{m/s}^2 = 12.1\ \text{m/s}^2$$

SP 3.3 $\bar{v} = \frac{\text{total distance}}{\text{total time}}$ Total time $= T = t_1 + t_2$ $t_1 =$ time at 20 km/h

$$t_1 = \frac{20\text{ km}}{20\text{ km/h}} = 1\text{ h} \qquad \text{so } t_2 = T - 1$$

(a) If $\bar{v} = 10$ km/h, then

$$10\text{ km/h} = \frac{40\text{ km}}{1 + t_2} \qquad t_2 = 3\text{ h} \qquad \bar{v} = \frac{20\text{ km}}{3\text{ h}} = 6.7\text{ km/h}$$

(b) If $\bar{v} = 30$ km/h, then

$$30\text{ km/h} = \frac{40\text{ km}}{1 + t_2} \qquad t_x = \frac{1}{3}\text{ h} \qquad \bar{v} = \frac{20\text{ km}}{1/3\,h} = 60\text{ km/h}$$

(c) As $\bar{v}_2 \rightarrow \infty$, $t_2 \rightarrow 0$ and

$$\bar{v}_{\max} = \frac{40\text{ km}}{1\text{ h}} = 40\text{ km/h}$$

SP 3.4 $y = bt^2$ $v = \frac{dy}{dt} = bt$ $a = \frac{dv}{dt} = b$

SP 3.5 $x = A\cos 2\pi\text{ ft}$ $v = \frac{dx}{dt} = -2\pi f\,A\sin 2\pi\text{ ft}$ $a = \frac{dv}{dt} = -(2\pi f)^2 A\cos 2\pi\text{ ft}$

SP 3.6 $y = y_0 + v_0t - \frac{1}{2}\,gt^2$

Let $y_0 = 0$, $v_0 = 0$, and $y = -54.5$ m at ground. Then

$$-54.5 = -\frac{1}{2}\,(9.8)t^2 \qquad t = 3.33\text{ s}$$

SP 3.7 $h = \frac{1}{2}\,gt^2$ $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(800\text{ ft})(0.305\text{ m})}{9.8\text{ m/s}^2}}\text{ m/ft} = 7.06\text{ s}$

$$v = v_0 - gt = 0 - (9.8\text{ m/s}^2)(7.06\text{ s}) = 69.2\text{ m/s (about 155 mi/h)}$$

SP 3.8 $v_0 = 60\text{ km/h} = 60\frac{1000\text{ m}}{3600\text{ s}} = 16.7\text{ m/s}$

(a) $v^2 = v_0^2 + 2ax = 0$ when stopped, so

$$x = -\frac{v_0^2}{2a}$$

$$= \frac{(16.7\text{ m/s})^2}{2(-0.5)(9.8\text{ m/s}^2)} = 28.3\text{ m}$$

(b) During the 0.20-s reaction time, the car travels a distance of

$$x_r = vt_r = (16.7\text{ m/s})\,(0.20\text{ s}) = 3.33\text{ m}$$

The total stopping distance is thus $28.3\text{ m} + 3.3\text{ m} = 31.6\text{ m}$.

SP 3.9 (a) In 1 s the first package falls a distance of $h = 1/2\ gt^2 = 1/2\ (9.8\ \mathrm{m/s^2})(1\ \mathrm{s})^2 = 4.9\ \mathrm{m}$.

(b) The first package hits the ground after time t_1, where $800 = 1/2\ gt_1^2$, so $t_1 = 12.8\ \mathrm{s}$. The second package thus falls for 11.8 s, dropping a distance of

$$h_2 = \tfrac{1}{2}\ gt_2^2 = \tfrac{1}{2}\ (9.8\ \mathrm{m/s^2})(11.8\ \mathrm{s})^2 \qquad \mathrm{h}_2 = 680\ \mathrm{m}$$

Thus the second package is $800\ \mathrm{m} - 680\ \mathrm{m} = 120\ \mathrm{m}$ above ground when the first package hits.

SP 3.10 The distance the baseball falls is $h = 1/2\ gt^2$. The distance the golf ball rises is $y = v_0 t\ - 1/2\ gt^2$.

$$h + y = 18 \qquad \text{so } \tfrac{1}{2}\ gt^2 + v_0 t - \tfrac{1}{2}\ gt^2 = 18$$

$$v_0 = 20\ \mathrm{m/s} \qquad \text{so } t = \frac{18}{v_0} = \frac{18\ \mathrm{m}}{20\ \mathrm{m/s}} = 0.90\ \mathrm{s}$$

$$h = \tfrac{1}{2}\ gt^2 = \tfrac{1}{2}\ (4.9\ \mathrm{m/s^2})(0.90\ \mathrm{s})^2 = 3.97\ \mathrm{m}$$

SP 3.11 (a) $v = v_0 + at = 0$,

$$t = -\frac{v_0 a}{a} = -\frac{100\ \mathrm{m/s}}{(-5.2\ \mathrm{m/s^2})} = 19.2\ \mathrm{s}$$

(b) $v^2 = v_0^2 + 2ax = 0$,

$$x = -\frac{v_0^2}{2a} = -\frac{(100\ \mathrm{m/s})^2}{(2)(-5.2\ \mathrm{m/s^2})} = 962\ \mathrm{m}$$

SP 3.12 To fall a distance $h = 12\ \mathrm{m}$ from rest requires time t, where

$$h = \tfrac{1}{2}\ gt^2 \qquad t^2 = \frac{2h}{g} = \frac{(2)(12\ \mathrm{m})}{9.8\ \mathrm{m/s^2}} \qquad t = 1.56\ \mathrm{s}$$

In 1.56 s the truck moves a distance of $x = vt$:

$$x = (80 \tfrac{1000\ \mathrm{m}}{3600\ \mathrm{s}})(1.56\ \mathrm{s}) = 34.8\ \mathrm{m}$$

The stunt man should drop when the truck is about 35 m away.

SP 3.13 The time for the first ball to reach the ground is t_1, $y = y_0 + v_0 t - 1/2\ gt^2$. Let $y = 0$ at starting point, so $y = -h = -24\ \mathrm{m}$ at the ground.

$$v_0 = 12\ \mathrm{m/s} \qquad -24 = 0 + 12t - \tfrac{1}{2}(9.8)t^2 \qquad 4.9t^2 - 12t - 24 = 0$$

$$t = \frac{12 \pm \sqrt{(12)^2 - 4(4.9)(-24)}}{(2)(4.9)} = 3.75\ \mathrm{s} \quad \text{or} \quad -1.30\ \mathrm{s}$$

The ball was thrown at $t = 0$, so it hits the ground at a later time, at $t = 3.75\ \mathrm{s}$. The ball dropped from rest will require time t_2 to reach the ground, where

$$h = \tfrac{1}{2}\ gt_2^2 \qquad t_2^2 = \frac{2h}{g} = \frac{2(24\ \mathrm{m})}{9.8\ \mathrm{m/s^2}} \qquad t_2 = 2.21\ \mathrm{s}$$

Thus the second ball should be dropped a time Δt later, where

$$\Delta t = t - t_2 = 3.75\,\mathrm{s} - 2.21\,\mathrm{s} = 1.54\,\mathrm{s}$$

SP 3.14 The object starts at rest, so $v_0 = 0$, $v_1^2 = v_0^2 - 2gy = 0 - 2g$. y = distance dropped $= -h$, so $v_1^2 = 2gh$. For the stopping process, the initial velocity is v_1, so $v^2 = v_1^2 + 2a(-d)$. Note that the stopping force, and hence the stopping acceleration, is upward and positive. When stopped, $v^2 = 0 = v_1^2 - 2\,ad$ and $0 = 2gh - 2\,ad$. So $a/g = h/d$.

SP 3.15 $v_1 = 50\,\mathrm{km/h} = 13.9\,\mathrm{m/s}$ $v_2 = 100\ \mathrm{km/h} = 27.8\ \mathrm{m/s}$

$$v^2 = v_0^2 + 2\,ax = 0$$

when stopped. Thus

$$x = -\frac{v_0^2}{2a}$$

For $v_0 = v_1$

$$x_1 = \frac{-(13.9\,\mathrm{m/s})^2}{2(-4.9\,\mathrm{m/s^2})} = 19.7\,\mathrm{m}$$

For $v_0 = v_2$

$$x_2 = \frac{-(27.8\,\mathrm{m/s})^2}{2(-4.9\,\mathrm{m/s^2})} = 78.7\,\mathrm{m}$$

Doubling the speed increases the stopping distance by a factor of 4. Speed kills!

SP 3.16 Suppose the binoculars were dropped from an altitude h. They would strike the ground after time t_1, where

$$h = \tfrac{1}{2}\,gt_1^2 \qquad (i)$$

The time to reach a point 12 m above the ground is t_2, where

$$h - 12 = \tfrac{1}{2}\,gt_2^2 \qquad (ii)$$

Also, we are told

$$t_1 - t_2 = 0.38\,x \qquad (iii)$$

We thus have three simultaneous equations for the unknowns t_1, t_2, and h. Substitute *(i)* and *(iii)* into *(ii)*:

$$\tfrac{1}{2}\,gt_1^2 - 12 = \tfrac{1}{2}\,g\,(t_1 - 0.38)^2$$

Solve

$$t_1 = 3.41\,\mathrm{s}$$

$$h = \tfrac{1}{2}\,gt_1^2 = \tfrac{1}{2}\,(9.8\,\mathrm{m/s^2})(3.41\,\mathrm{s})^2 = 57.1\,\mathrm{m}$$

SP 3.17 In 1 year the earth travels a distance $2\pi r$, so

$$v = \frac{2\pi r}{T} = \frac{(2\pi)(1.5\times 10^8\,\mathrm{km})}{(365)(24\,\mathrm{h})} = 108{,}000\ \mathrm{km/h}$$

SP 3.18 Stopping distance is

$$x_s = v_0 t_R + x_1 \qquad v_0 = 100 \text{ km/h} = 27.8 \text{ m/s}$$

where

$$v^2 = v_0^2 + 2\,a x_1 = 0$$

so

$$x_s = v_0 t_R + \frac{v_0^2}{-2a} = (27.8 \text{ m/s})(0.70 \text{ s}) + \frac{1}{-2}\,\frac{(27.8 \text{ m/s})^2}{(-0.6)(9.8 \text{ m/s}^2)}$$

$$= 19.5 \text{ m} + 65.7 \text{ m} = 85.2 \text{ m}$$

The sheep is wiped out. The speed when $x = 32$ m (where the sheep is) is given by

$$v^2 = v_0^2 + 2\,ax \qquad v^2 = (27.8 \text{ m/s})^2 + 2(-0.6)(9.8 \text{ m/s}^2)(32 \text{ m}) \qquad v = 19.9 \text{ m/s}$$

SP 3.19 Let $x = 2400$ km = one-half of the trip:

$$x_1 = v_0 t_1 + \tfrac{1}{2}\,a t_1^2 \qquad v_0 = 0$$

so

$$t_1 = \sqrt{\frac{2x_1}{a}} = \sqrt{\frac{(2)(2400 \times 10^3 \text{ m})}{(0.4)(9.9 \text{ m/s}^2)}}$$

$$= 1106 \text{ s}$$

$$\text{Total trip time} = T = 2t_1 = 2212 \text{ s} \cong 37 \text{ min}$$

$$\text{Speed } v = v_0 + at = 0 + (0.4)(9.8 \text{ m/s}^2)(1106 \text{ s}) = 4340 \text{ m/s} \simeq 9700 \text{ mi/h}$$

SP 3.20 $v^2 = v_0^2 + 2\,ax = 0$ if the sports car stopped at the end of the skid marks. Thus

$$v_0^2 = -2\,ax = -2(-0.9)(9.8 \text{ m/s}^2)(240 \text{ m}) \qquad v_0 = 65 \text{ m/s} \simeq 146 \text{ mi/h}$$

Chapter 4

Motion in a Plane

Having described in the previous chapter how to define motion in one dimension, I will now extend these ideas to include motion in two or three dimensions. However, in many important problems (for example, projectiles, planetary orbits, oscillations of a pendulum), the motion is limited to a plane, so I will limit the discussion to this case. The basic features of three-dimensional motion readily follow from this treatment.

4.1 POSITION, VELOCITY, AND ACCELERATION

For motion in one dimension I was able to describe vector properties simply by assigning a plus or minus sign to them. Now we must use more explicit notation to make clear the vector property. We specify the position of a particle by the **position vector r**. As the particle moves, **r** changes, as illustrated in Figure 4.1. If at time t_1 the position vector is $\mathbf{r}_1$ and at time t_2 it is $\mathbf{r}_2$, the **displacement vector** for this time interval is defined as $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

The position, velocity, and acceleration vectors for a particle moving in the x-y plane are:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{4.1}$$

$$\mathbf{v} = \tfrac{d}{dt}\mathbf{r}(t) = \tfrac{d}{dt}(x\mathbf{i} + y\mathbf{j}) = \tfrac{dx}{dt}\mathbf{i} + \tfrac{dy}{dt}\mathbf{j} \tag{4.2}$$

$$= v_x\mathbf{i} + v_y\mathbf{j} = \mathbf{v}_x + \mathbf{v}_y$$

$$v_x = \tfrac{dx}{dt} \qquad v_y = \tfrac{dy}{dt} \qquad \mathbf{v}_x = \tfrac{dx}{dt}\mathbf{i} \qquad \mathbf{v}_y = \tfrac{dy}{dt}\mathbf{j}$$

$$\mathbf{a} = \tfrac{dv_x}{dt}\mathbf{i} + \tfrac{dv_y}{dt}\mathbf{j} = a_x\mathbf{i} + a_y\mathbf{j} \tag{4.3}$$

$$a_x = \tfrac{dv_x}{dt} = \tfrac{d^2x}{dt^2} \qquad a_y = \tfrac{dv_y}{dt} = \tfrac{d^2y}{dt^2}$$

The magnitudes of these vectors are:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2} \qquad v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \qquad a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$$

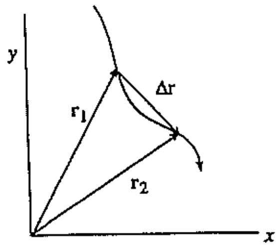


Figure 4.1

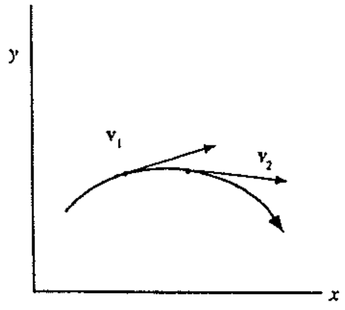


Figure 4.2

The velocity vector is directed tangent to the path of the particle (see Figure 4.2). The acceleration vector can be oriented in any direction, depending on what is happening. The component of the acceleration vector that is parallel or antiparallel to **v** (that is, along the line tangent to the path) is called the **tangential acceleration**. A more descriptive name is "speeding up or slowing down acceleration." This is the kind of acceleration that measures changes in speed, as studied in the previous chapter.

4.2 CONSTANT ACCELERATION

The equations obtained previously for motion in one dimension with constant acceleration apply here as well. *Caution: For problems involving motion in two dimensions, it is very important that you use subscripts to indicate if you are dealing with quantities associated with x or with y. Failure to do this is a common source of error.* If acceleration is constant, both a_x and a_y are constant.

$$\boxed{\begin{array}{ll} x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 & y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ v_x = v_{0x} + a_x t & v_y = v_{0y} + a_y t \\ v_x^2 = v_{0x}^2 + 2a_x x & v_y^2 = v_{0y}^2 + 2a_y y \end{array}} \qquad (4.4)$$

These equations can be written in compact vector form.

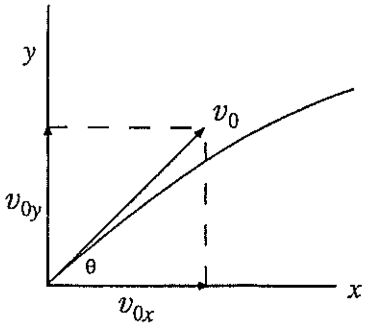
$$\mathbf{r} = \mathbf{r} + \mathbf{v}_0 t + \mathbf{a}t^2 \qquad \mathbf{v} = \mathbf{v}_0 + \mathbf{a}t \qquad \mathbf{a} = \text{constant} \qquad (4.5)$$

In using these equations, take motion to the right as positive for x and motion upward as positive for y. The x axis and the y axis are normally horizontal and vertical, but any two perpendicular axes can be used, and equations of the above form will apply.

4.3 PROJECTILES

Consider an object that flies through the air subject to no force other than gravity and air resistance. The gravity force causes a constant downward acceleration of magnitude $g = 9.80\ \text{m/s}^2$. As a first approximation, I will neglect the effects of air and of variations in g. I assume the earth is flat over the horizontal range of the projectiles. I neglect air effects mainly because they are complicated to include, not because they are insignificant in all cases. Despite these simplifying assumptions, we can still obtain a fairly good description of projectile motion. The path of a projectile is called its **trajectory.**

If air resistance is neglected, there is then no acceleration in the horizontal direction, and $a_x = 0$. The acceleration in the y direction is that due to gravity. It is constant and directed downward, so $a_y = -g$. It is convenient to choose $x_0 = 0$ and $y_0 = 0$ (that is, place the origin at the point where the projectile starts its motion). Further, we typically are concerned with the initial speed v_0 of the projectile. If the projectile is launched at an angle θ above horizontal, the initial velocity in the x direction and the initial velocity in the y direction can be expressed in terms of θ and v_0 using trigonometry.



$$\boxed{v_{0x} = v_0 \cos\,\theta \quad \text{and} \quad v_{0y} = v_0 \sin\,\theta} \tag{4.6}$$

Equation 4.3 thus becomes

$$\boxed{\begin{array}{ll} a_x = 0 & a_y = -g \\ v_x = v_0 \cos\,\theta = \text{constant} & v_y = v_0 \sin\theta - gt \\ x = (v_0 \cos\theta)t & y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \end{array}} \tag{4.7}$$

From the equation for x we can obtain $t = x/v_0 \cos\theta$. Substitute this in the equation for y and obtain

$$y = (\tan\theta)\,x - \left(\frac{g}{2v_0^2 \cos{}^2\theta}\right) x^2 \tag{4.8}$$

This is the equation of a parabola that passes through the origin.

A key feature of projectile motion is that **the horizontal motion is independent of the vertical motion.** Thus a projectile moves at a constant speed in the horizontal direction, independent of its vertical motion. This is illustrated in Figure 4.3.

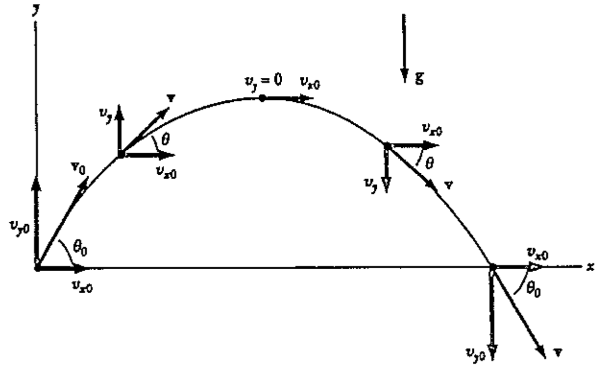


Figure 4.3

We can gain insight into the meaning of Eq. 4.6 by viewing projectile motion in this way: First, if there were no gravity force and downward acceleration, in time t the projectile would move a distance v_0t in a straight inclined line. If now we imagine gravity "turned on," the effect would be to make the projectile fall away from the straight line path by a distance 1/2 gt^2. The superposition of these two effects results in the parabolic path observed. See Figure 4.4.

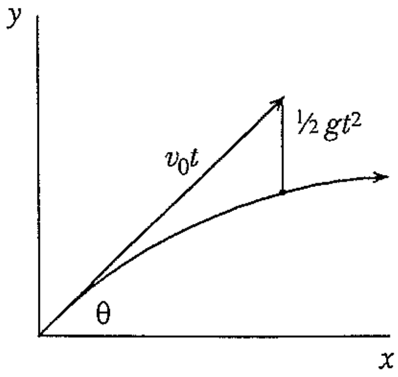
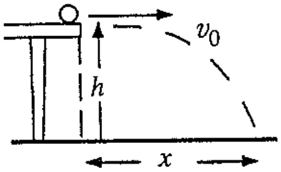


Figure 4.4

Problem 4.1 A ball rolls off a table 80 cm high with speed of 2.4 m/s. How far will it travel horizontally before striking the ground?

Solution Here is a good problem-solving strategy to follow for any challenging problem. First draw a schematic picture so that you are very clear on what is happening. In your drawing, indicate known quantities, and label unknown quantities with appropriate symbols. Next decide what principles or laws you will apply, and write them down in equation form. Manipulate the equations to obtain the desired result, and finally, substitute numerical values. I solve this problem by reasoning as follows: I know that the horizontal velocity is constant (2.4 m/s), so I reason that if I knew the time in the air, I could find the horizontal distance since $x = v_xt$. Since falling is independent of moving sideways, I can find the time to fall down 80 cm, starting with zero vertical velocity. Thus I can find the time, and hence the horizontal distance. Notice that the problem statement did not ask for the time to fall. You have to realize on your own that the time must be found. This is typical of many "two-step" problems and is illustrative of many real-life problems such as those encountered in diagnosing a disease or designing a traffic control system. Finding the time of flight is involved in a majority of projectile problems. Here we recognize



$$v_{0x} = 2.24\,\text{m/s}$$

$$v_{0y} = 0 \quad \text{(The ball rolls off horizontally.)}$$

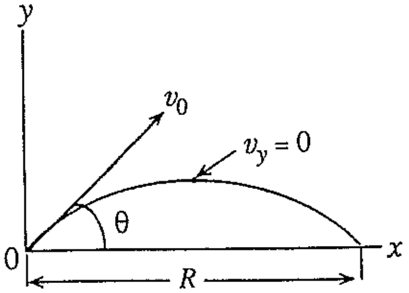
The time to fall a distance h with zero initial vertical velocity is given by $h = 1/2\ gt^2$, so $t = \sqrt{2h/g}$. The horizontal distance x is thus

$$\begin{aligned} v_{0x}t &= v_{0x}2\,\frac{h}{g} = (2.4\ \text{m/s})\sqrt{\frac{2(0.80\,\text{m})}{(9.8\,\text{m/s}^2)}} \\ &= 0.97\,\text{m} \end{aligned}$$

Problem 4.2 A golf ball is hit at an angle of 30° above horizontal with a speed of 44 m/s. How high does it rise, how long is it in the air, and how far does it travel horizontally?

Solution The components of the initial velocity are:

$$\begin{aligned} v_{0x} &= v_0 \cos\theta = 44 \cos 30 = 38.1\,\text{m/s} \\ v_{0y} &= v_0 \sin\theta = 44 \sin 30 = 22\,\text{m/s} \end{aligned}$$



At the highest point, $v_y = v_0 \sin\theta - gt = 0$, so

$$t = \frac{v_0 \sin\theta}{g} = \frac{(44\ \text{m/s})(\sin 30°)}{9.8\,\text{m/s}^2} = 2.24\,\text{s}$$

$$\begin{aligned} y &= (v_o \sin\theta)\,t - \tfrac{1}{2}gt^2 \\ &= (44\,\text{m/s})(\sin 30°)(2.24\,\text{s}) \\ &\quad - (0.5)(9.8\,\text{m/s}^2)(2.24\,\text{s})^2 \\ &= 24.7\,\text{m} \end{aligned}$$

Another way of obtaining this answer is to use $v_y^2 = v_{0y}^2 - 2gy$. At the highest point $v = 0$, so $y = v_{0y}^2/2g = (22\,\text{m/s})^2/2(9.8\,\text{m/s}^2) = 24.7\,\text{m}$.

When the ball returns to ground level, $y = 0$, so T, the total time in the air, can be found from $y = (v_0\ \sin\theta)\ T - 1/2\ gT^2 = 0$. Thus, $T = 2(v_0 \sin\theta)/g = 2(44\,\text{m/s})$ (sin $30°)/(9.8\ \text{m/s}^2) = 4.49\,\text{s}$. Thus the horizontal range ($x = R$ when $y = 0$) is

$$R = (v_0 \cos\theta)T = \frac{2v_0^2\,(\sin\theta\cos\theta)}{g} \tag{4.9}$$

Since $2\sin\theta\cos\theta = \sin 2\theta$, we can write

$$R = \frac{(v_0^2 \sin 2\theta)}{g} \tag{4.10}$$

Thus $$R = \frac{(44\,\text{m/s})^2(\sin 60^\circ)}{9.8\,\text{m/s}^2} = 171\,\text{m}$$

From Eq. 4.9 we can see that the maximum range, for a given initial velocity, results when $\sin 2\theta$ is a maximum. Sin 2θ is biggest when $2\theta = 90^\circ$, or $\theta = 45$. If air effects are taken into account, it turns out that the maximum range occurs for a slightly lower angle of elevation.

Inspection of Eq. 4.8 shows that two initial angles of elevation α and β will result in the same range provided that $\alpha + \beta = 90^\circ$. This is so because if α and β are complementary angles (they add to 90°), $\sin\alpha = \cos\beta$ and $\cos\alpha = \sin\beta$. Two such angles differ from 45° by the same amount, for example, 50° and 40°, 65° and 25°, and 71° and 19° (see Figure 4.5).

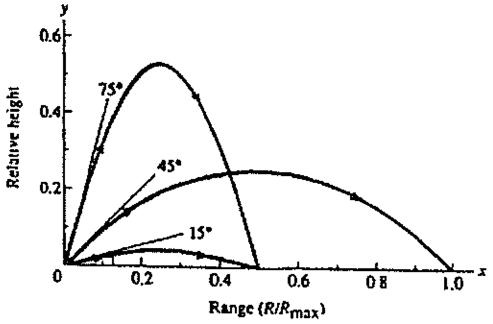


Figure 4.5

Problem 4.3 I've sometimes wondered if it is possible to throw a baseball high enough to hit the roof of a domed stadium like the King Dome in Seattle. I've seen a center fielder throw all the way from center field to home plate on the fly. Assume such a throw traveled 120 m horizontally and was thrown at an angle of elevation to maximize the range. How high would the ball go if thrown straight up?

Solution For the maximum range, $\theta = 45^\circ$. From Eq. 4.9 I can determine v_0, given $R = 120$ m and $\theta = 45^\circ$. Thus

$$120\,\text{m} = \frac{v_0^2 \sin 2(45^\circ)}{9.8\,\text{m/s}} \qquad v_0 = 34.3\,\text{m/s}$$

If thrown straight up, $\theta = 90^\circ$ and $v_{0y} = v_0 = 34.3\,\text{m/s}$. $v_y^2 = v_{0y}^2 - 2gy = 0$ at the highest point, so

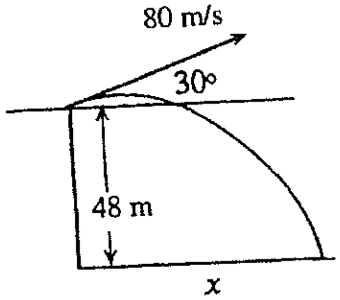
$$y = \frac{v_0^2}{2g}$$

$$= \frac{(34.3\,\text{m/s})^2}{2(9.8\,\text{m/s}^2)} = 60\,\text{m}$$

Incidentally, notice that this result, maximum range = 2 × maximum height, is true generally.

Problem 4.4 An archer standing on a cliff 48 m above the level field below shoots an arrow at an angle of 30° above horizontal with a speed of 80 m/s. How far from the base of the cliff will the arrow land?

Solution I could use Eq. 4.7 to find x, since we are given $y = -48$ m, $\theta = 30°$, and $v_0 = 80\,\text{m/s}$. This will require solving a quadratic equation. An alternate (but equivalent) approach is to find the time in the air and then determine the range from $x = (v_0 \cos\theta)t$. Thus



$$y = (v_0 \sin\theta)t - \tfrac{1}{2}\,gt^2 - 48$$

$$= (80 \sin 30°)t - (0.5)(9.8)t^2$$

$$4.9t^2 - 40t - 48 = 0$$

$$t = \frac{40t\sqrt{(40)^2-4(4.9)(-48)}}{2(4.9)} = 8.40\,\text{s or } -0.24\,\text{s}$$

$$x = (v_0 \cos\theta)t = (80\,\text{m/s})(\cos 30°)(8.4\,\text{s}) = 582\,\text{m}$$

4.4 UNIFORM CIRCULAR MOTION

An object that moves in a circle with constant speed is in **uniform circular motion.** Although the magnitude of the velocity vector (the speed) is constant, the direction of the velocity is changing. Recall that acceleration measures the rate of change of velocity. In the previous chapter I discussed acceleration associated with changes in speed ("tangential" acceleration). Here we consider acceleration associated with a change in direction of the velocity vector. This is what I would call "turning acceleration" but what other books would call **centripetal acceleration** or **radial acceleration**. Figure 4.6 illustrates how the position vector **r** and the velocity vector **v** change as a particle moves around a circle. The velocity vector **v** is tangent to the circle. Think of **v** and **r** as being rigidly joined together, like the sides of a carpenter's square. When **r** moves through an angle θ, **v** moves through the same angle.

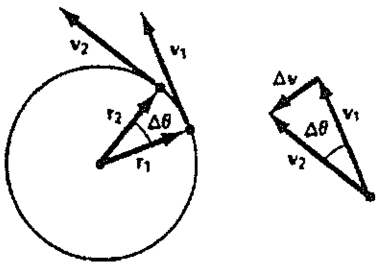


Figure 4.6

Suppose the particle moves through a very small angle $\Delta\theta$. Slide the two velocity vectors together so that they form a small isosceles triangle with sides v_1, v_2, and Δv. The angle between the two sides of length v is $\Delta\theta$. Now consider the isosceles triangle formed by $\mathbf{r}_1$ and $\mathbf{r}_2$. If $\Delta\theta$ is very small, the short side of this triangle is approximately equal to the arc length Δs subtended by $\Delta\theta$. If $\Delta\theta$ is measured in radians, $\Delta s = r\Delta\theta$. These two isosceles triangles are similar triangles, so their sides are in the same ratio. Thus

$$\frac{\Delta v}{v} = \frac{\Delta s}{r} = \frac{v\,\Delta t}{r}$$

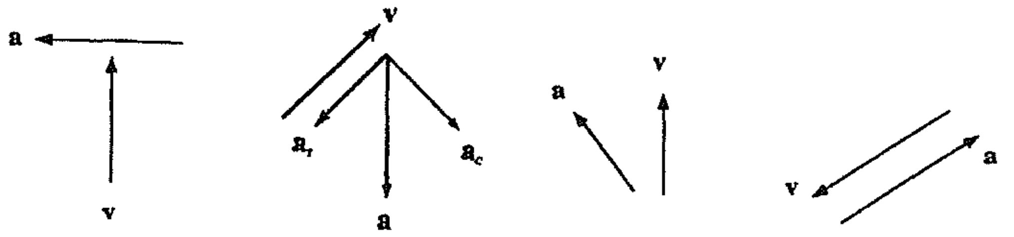
where $\Delta s = v\,\Delta t$

so $\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$

As $\Delta t \to 0$, then $a_c = \frac{\Delta v}{\Delta t}$ so $\boxed{a_c = \frac{v^2}{r}}$ *(4.11)*

This turning acceleration is called **centripetal acceleration** because this acceleration vector is directed "toward the center." One can see this, since $\mathbf{a} \simeq \Delta\mathbf{v}/t$ and $\Delta\mathbf{v}$ points inward toward the center of curvature. Note that the particle does not have to move in a full circle to experience centripetal acceleration. Any arc can be thought of as a small part of a circle. Remember, if an object turns left, it accelerates left. If it turns right, it accelerates right. Incidentally, you may have heard the expression "centrifugal acceleration." Forget you ever heard this term, and never, never use it. It will only confuse you. Long ago this term was used in connection with a confusing notion of fictitious forces.

An object can experience both centripetal (turning) acceleration and tangential (speeding up or slowing down) acceleration. In Figure 4.7 are shown some possible combinations for $\mathbf{v}$ and $\mathbf{a}$ for a moving car. To understand the acceleration, resolve it into components parallel to $\mathbf{v}$ and perpendicular to $\mathbf{v}$. To tell if the car is turning right or left, imagine that you are the driver sitting with the velocity vector directed straight ahead in front of you. A forward component of acceleration means speeding up.



Turning Left at Constant Speed

Turning Right and Slowing Down

Turning Left and Speeding Up

Slowing Down, No Turning

Figure 4.7

Problem 4.5 A military jet fighter plane flying at 180 m/s pulls out of a vertical dive by turning upward along a circular path of radius 860 m. What is the acceleration of the plane? Express the acceleration as a multiple of g.

Solution $$a = \frac{v^2}{r} = \frac{(180\,\text{m/s})^2}{860\,\text{m}} = 37.7\,\text{m/s}^2 = \frac{37.7}{9.8g} = 3.8g$$

4.5 RELATIVE MOTION

To describe motion, we must refer it to a **frame of reference**. Often we use a reference frame attached to the surface of the earth or to the floor of the room. On a moving train, we might use the floor of the train car as the reference frame. If a person in a train moving at constant velocity drops a pencil, he will see it fall straight down. A person on the ground will see the pencil drop along a parabolic path. We frequently encounter such problems in connection with navigation. To solve such problems, label the tip of a velocity vector with a symbol representing the moving object. Label the tail of the velocity vector with a symbol representing the reference frame. To see how this works, consider the following example:

Problem 4.6 A person P walks at a speed of 1.5 m/s on a moving sidewalk SW in an airport terminal. The sidewalk moves at 0.8 m/s. How fast is the person moving with respect to the earth E?

Solution Draw to scale the velocity vector for the person with respect to the sidewalk, $v_{PSW} = 1.5$ m/s. Next draw the velocity vector for the sidewalk with respect to the earth, $v_{SWE} = 0.8\,\text{m/s}$.

Now slide these vectors parallel to themselves so that matching symbols are superimposed.

The vector representing the velocity of the person P with respect to the earth E is drawn from E to P, as shown. From the diagram we see $v_{PE} = 1.5\,\text{m/s} + 0.8\,\text{m/s} = 2.3\,\text{m/s}$.

The usefulness of this technique is illustrated by the following, more complicated problem.

Problem 4.7 A river flows due east at 5 km/h. A motorboat can move through the water at 12 km/h. (a) If the boat heads due north across the river, what will be the direction and magnitude of its velocity with respect to the earth? (b) In what direction should the boat head if it is to travel due north across the river? What will its speed with respect to earth then be?

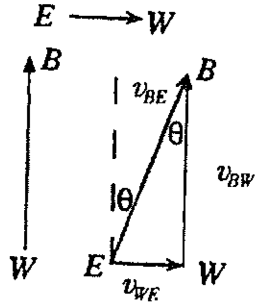
Solution Here are the velocity vectors for the water W with respect to earth E and for the boat B with respect to the water. Slide them together so that the W's touch. The velocity of the boat with respect to the earth is the vector drawn from E to B.

(a) Using trig, we find

$$v_{BE}^2 = (12\,\text{km/h})^2 + (5\,\text{km/h})^2$$

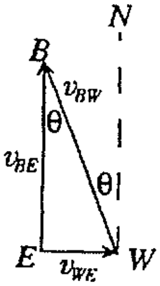
$$v_{BE} = 13\,\text{km/h}$$

$$\theta = \tan^{-1}\frac{5}{12} = 22.6°\ \text{E of N}$$



The boat will travel 22.6° east of north.

(b)



$$v_{BE}^2 + v_{WE}^2 = v_{BW}^2$$

$$v_{BE}^2 = (12\,\text{km/h})^2 - (5\,\text{km/h})^2$$

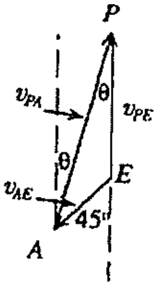
$$v_{be} = 10.9\,\text{km/h}$$

$$\theta = \sin^{-1}\frac{5}{12} = 24.6°\ \text{W of N}$$

Problem 4.8 A pilot with an airspeed (speed with respect to air) of 120 km/h wishes to fly due north. A 40-km/h wind is blowing from the northeast. In what direction should she head, and what will be her ground speed (speed with respect to the ground)?

Solution

$$(v_{PE} + v_{AE}\cos 45°)^2 + (v_{AE}\sin 45°)^2 = v_{PA}^2$$



$$(v_{PE} + 40\cos 45°)^2 = (120)^2 - (40\sin 45°)^2$$

$$v_{PE} = 88.3\,\text{km/h}$$

$$\theta = \sin^{-1}\frac{v_{AE}\sin 45°}{v_{PA}} = \sin^{-1}\frac{(40\sin 45°)^2}{120} = 13.6°\,\text{E of N}$$

The plane should head 13.6° east of north.

4.6 SUMMARY OF KEY EQUATIONS

For constant acceleration in x and y directions,

$$x = x_0 + v_{0x}t + \tfrac{1}{2}\, a_x t^2 \qquad y = y_0 + v_{0y}t + \tfrac{1}{2}\, a_y t^2$$

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t$$

$$v_x^2 = v_{0x}^2 + 2a_x x \qquad v_y^2 = v_{0y}^2 + 2a_y y$$

For projectiles,

$$\begin{array}{lcl} v_{0x} = v_0 \sin\theta & \text{and} & v_{0y} = v_0 \cos\theta \\ a_x = 0 & & a_y = -g \\ v_x = v_0 \cos\theta = \text{constant} & & v_y = v_0 \sin\theta - gt \\ x = (v_0 \cos\theta)\, t & & y = (v_0 \sin\theta)t - \tfrac{1}{2}\, gt^2 \end{array}$$

The equation of the path is a parabola.

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 \cos{2\theta}}\right) x^2$$

The time in the air is

$$T = \frac{2v_0 \sin\ \theta}{g}$$

The horizontal range is a maximum for $\theta = 45°$, where

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

The maximum height is

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

For uniform circular motion the inward centripetal (radial) acceleration is

$$a_c = \frac{v^2}{r}$$

Supplementary Problems

SP 4.1 An unidentified naval vessel is tracked by the Navistar Global Positioning System. With respect to a coordinate origin (0, 0) fixed at a lighthouse beacon, the position of the vessel is found to be $x_1 = 2.0$ km west, $y_1 = 1.6$ km south at $t_1 = 0.30$ h and $x_2 = 6.4$ km west, $y_2 = 6.5$ km north at $t_2 = 0.60$ h. Using east-west as the x axis and north-south as the y axis, determine the average velocity in terms of its components. What are the direction and magnitude of the average velocity in kilometers per hour?

SP 4.2 The track of a cosmic ray particle in a photographic emulsion is found empirically to be described by the expression $\mathbf{r} = (3t^3 - 6t)\,\mathbf{i} + (5 - 8t^4)\,\mathbf{j}$. Determine the velocity and acceleration.

SP 4.3 A sandbag is dropped from rest from a hot air balloon at an altitude of 124 m. A horizontal wind is blowing, and the wind gives the sandbag a constant horizontal acceleration of 1.10 m/s^2. (a) Show that the path of the sandbag is a straight line. (b) How long does it take to hit the ground? (c) With what speed does it hit the ground?

SP 4.4 A charged dust particle generated in an environmental study of smoke stack efficiency moves through a velocity selection device with constant acceleration $\mathbf{a} = 4\mathbf{j}$ m/s^2 with an initial velocity of $\mathbf{v} = 6\mathbf{i}$ m/s. Determine the speed and position of the particle when $t = 4$ s.

SP 4.5 An artillery shell is fired so that its horizontal range is twice its maximum height. At what angle is it fired?

SP 4.6 A motorcycle rider wants to jump a ditch 4 m wide. He leaves one side on a ramp that slopes up at 20° above horizontal. He lands at the same elevation at which he took off. His front wheel leaves the ground 1 m before the edge of the ditch and comes down 2 m past the far side of the ditch. What minimum take-off speed is required?

SP 4.7 During World War I the Germans reportedly bombarded Paris from about 50 km away with a long-barreled cannon called the Big Bertha. Iraq was suspected of building a similar weapon to launch nuclear bombshells on Israel in 1992. Neglecting air resistance, estimate the muzzle velocity needed by the Big Bertha. Muzzle velocity is the initial speed at which the shell leaves the gun.

SP 4.8 Migrating salmon are known to make prodigious leaps when swimming up rivers. The highest recorded jump by such a fish was 3.45 m upwards. Assuming the fish took off at an angle of 45° horizontal, with what speed did the fish leave the water?

SP 4.9 A rifle bullet is fired with a speed of 280 m/s up a plane surface that is inclined at 30° above horizontal. The bullet is fired at an initial angle of elevation of 45° above horizontal (that is, at 15° above the plane surface). How far up the plane does it land? (Problems like this are discussed in I. R. Lapides, *Amer. Jour. Phys.*, **51** (1983), p. 806, and H. A. Buckmaster, *Amer. Jour. Phys.*, **53** (1985), p. 638.

SP 4.10 A girl throws a ball from a balcony. When the ball strikes the ground, its path makes an angle θ with the ground. What is the minimum value of θ?

SP 4.11 A high-powered 7-mm Remington magnum rifle fires a bullet with a velocity of 900 m/s on a rifle range. Neglect air resistance. (a) Calculate the distance h such a bullet will drop at a range of 200 m when fired horizontally. (b) To compensate for the drop of the bullet, when the telescope sight is pointed right at the target, the barrel of the gun is aligned to be slanted slightly upward, pointed a distance h above the target. The downward fall due to gravity then makes the bullet strike the target as desired. Suppose, however, such a rifle is fired uphill at a target 200 m distant. If the upward slope of the hill is 45°, should you aim above or below the target, and by how much? What should you do when shooting on a *downhill* slope at 45° below horizontal?

SP 4.12 The radius of the earth is about 6370 km. Calculate the centripetal acceleration of a person at the equator.

SP 4.13 An electric fan rotates at 800 revolutions per minute (rev/min). Consider a point on the blade a distance of 0.16 m from the axis. Calculate the speed of this point and its centripetal acceleration.

SP 4.14 The fastest train in the United States is the Amtrak X2000, with a top speed of about 70 m/s (about 157 mi/h). Train passengers find the ride slightly uncomfortable if their acceleration exceeds $0.05g$. (a) What is the smallest radius of curvature for a bend in the track that can be tolerated within this limit?

(b) If the train had to go around a curve of radius 1.20 km, to what speed would the train have to be slowed in order not to exceed an acceleration of $0.05g$?

SP 4.15 A race car driver increases her speed uniformly from 60 to 66 m/s in a period of 4.0 s while rounding a curve of radius 660 m. At the instant when her speed is 63 m/s, what is the magnitude of her tangential acceleration? Of her centripetal acceleration? Of her total acceleration?

SP 4.16 The pilot of a passenger jet with an airspeed of 700 km/h wishes to fly 1400 km due north. To move to the north, the pilot finds he must fly in a direction pointed 10° west of north. If the flight requires 1 h 54 min, what is the wind velocity?

SP 4.17 A passenger in a car traveling 11 m/s (about 25 mi/h) notices that raindrops outside seem to be falling at an angle of about 60° with vertical. From this data, what would you estimate the speed of the falling raindrops to be? (Incidentally, because of air resistance, the rain is falling with constant velocity by the time it approaches the ground.)

SP 4.18 A moving sidewalk in an airport terminal moves at 1.20 m/s. It is 80 m long. A man steps on the sidewalk and walks to the other end at a speed of 0.8 m/s with respect to the sidewalk. How long does it take him to reach the other end?

SP 4.19 A river 86 m wide flows due west at 2.2 m/s. A man in a boat heads due south with respect to the water, moving at a speed of 4.8 m/s through the water. How long does it take him to cross the river? How far west of his starting point does he land?

SP 4.20 Kate can swim 0.90 m/s. She tries to swim across a river that is flowing 1.80 m/s. She heads in a direction that will minimize her drift downstream, but she still lands 120 m downstream from the point directly across from where she started. In what direction did she swim, and how wide was the river?

SP 4.21 The currents in the Strait of Juan de Fuca at the entrance to Puget Sound can be very swift. Travel there in a small fishing boat can be hazardous. Suppose the current is coming in from the open sea at a speed of 23 km/h, directed due east. A fisherman wants to travel north from Port Angeles to Victoria, British Columbia, a distance of about 48 km. He needs to make the trip in 2 h 15 min, but he isn't sure if his boat is fast enough. What minimum speed would he need? (A boat's speed is measured with respect to the water it moves through.)

Solutions to Supplementary Problems

SP 4.1

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{(-6.4\,\mathbf{i} + 6.5\,\mathbf{j}) - (-2.0\,\mathbf{i} - 1.6\,\mathbf{j})}{0.60 - 0.30}$$

$$\mathbf{v} = -8.4\mathbf{i} + 4.9\mathbf{j}\ \text{km/h} \qquad v = \sqrt{(-8.4)^2 + (4.9)^2} = 9.72\ \text{km/h}$$

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \frac{4.9}{-8.4} = -30.3°$$

that is, 30.3° N of W

SP 4.2 $\mathbf{r} = (3t^2 - 6t)\mathbf{i} + (5 - 8t^4)\mathbf{j} \qquad \mathbf{v} = \frac{dr}{dt} = (6t - 6)\,\mathbf{i} - 32t^3\mathbf{j} \qquad \mathbf{a} = \frac{dv}{dt} = 6\,\mathbf{i} - 96t^2\mathbf{j}$

SP 4.3 $x = v_{0x}t + \frac{1}{2}a_xt^2 = \frac{1}{2}a_xt^2$ since $v_{0x} = 0 \quad y = v_{0y}t - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2$ since $v_{0y} = 0$

(a) Thus

$$t^2 = \frac{2x}{ax} = -\frac{2y}{g}, \qquad \text{or} \qquad y = -\frac{g}{a_x} \times \text{equation of a straight line}$$

(b) At ground,

$$y = -124\,\text{m}, \qquad \text{so } t^2 = -\frac{2(-124\,\text{m})}{9.8\,\text{m/s}^2} \qquad t = 5.03\,\text{s}$$

(c) $v_y = v_{0y} - gt = 0 - (9.8\ \text{m/s}^2)(5.03\ \text{s}) = -49.3\ \text{m/s}$

$v_x = v_{0x} + a_x t = 0 + (1.10\ \text{m/s}^2)(5.03\ \text{s}) = 5.53\ \text{m/s}$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(5.53\ \text{m/s}^2 + (-49.3\ \text{m/s})^2} = 49.6\ \text{m/s}$$

SP 4.4 $\mathbf{r} = \mathbf{v}_0 t + \frac{1}{2}\,\mathbf{a}t^2 = 6t\mathbf{i} + \frac{1}{2}(4\mathbf{j})t^2$ At $t = 4$ s, $\mathbf{r} = 24\,\mathbf{i} + 32\,\mathbf{j}$

SP 4.5 $v_y^2 = v_{0y}^2 - sgy = 0$ at the highest point. $v_{0y} = v_0 \sin\theta$, so

$$y = \frac{v_0^2 \sin\theta}{2g}$$

The range from Eq. 4.8 is

$$R = \frac{2v_0^2}{g}\sin\theta\cos\theta$$

Here $R = 2y$. Thus

$$\frac{2v_0^2}{g}\sin\theta\cos\theta = 2\,\frac{v_0^2\sin^2\theta}{2g}, \quad \cos\theta = \frac{1}{2}\sin\theta \qquad \tan\theta = 2, \quad \theta = 63.4\,^\circ$$

SP 4.6 From Eq. 4.9, the range is

$$R = \frac{v_0^2}{g}\sin 2\theta \qquad v_0^2 = \frac{Rg}{\sin 2\theta} \qquad v_0^2 = \frac{(7\,\text{m})(9.8\ \text{m/s}^2)}{\sin 2(20^\circ)} \qquad v_0 = 10.3\ \text{m/s}$$

SP 4.7 From Eq. 4.9, the range is

$$R = \frac{v_0^2}{g}\sin^2 2\theta$$

θ would have been chosen for the maximum range, so $2\theta = 90^\circ$.

$$v_0^2 = Rg = (50 \times 10^3\text{m})(9.8\,\text{m/s}^2) \qquad v_0 = 700\ \text{m/s}$$

SP 4.8 At the highest point $v_y = 0$, so $v_y^2 = (v_0 \sin\theta)^2 - 2gh = 0$.

$$(v_0\sin\theta)^2 = 2gh \qquad v_0^2 = \frac{2gh}{\sin\theta} = \frac{2(9.8\ \text{m/s}^2)(3.45\ \text{m})}{\sin 45^\circ} \qquad v_0 = 9.78\ \text{m/s}$$

SP 4.9 The equation of the inclined plane is

$$y = \tan 30^\circ \qquad x = \frac{1}{\sqrt{3}}x$$

The equation of the parabolic path is given in Eq. 4.7:

$$y = (\tan\ \theta)x - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2$$

The intersection of the parabola and the straight line occurs when

$$\frac{1}{\sqrt{3}}x = \tan\ (\tan\ \theta)x - \frac{g}{2v_0^2 \cos^2\theta}\,x^2$$

Substitute $\tan\ 45° = 1, \cos\ 45° = 1/\sqrt{2}$ and simplify. Find $x = v_0^2/g\ [1 - (1/\sqrt{3}\ )\ ]$. From the 30°-60°-90° triangle we see that $x = s\ \cos\ 30° = s\,(\sqrt{3}/2)$. So

$$s = \left(\frac{2}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\frac{v_0^2}{g} = \frac{2}{3}\ (\sqrt{3} - 1)\frac{v_0^2}{g} = \text{distance up the plane}$$

With $v_0 = 280\,\text{m/s}$, $s = 3.90\,\text{km}$. Another way of solving this problem is the following: If there were no gravity, the bullet would go straight to point P, a distance v_0t reached in time t. If gravity were "turned on," the bullet would also drop a distance $1/2\ gt^2$, to where it would hit the plane. By looking at the diagram, I see that

$$\begin{aligned} x &= h + H \\ &= \frac{1}{2}\ gt^2 + s\ \sin\ 30° \end{aligned} \qquad (i)$$

Also, $x = v_0t\cos\ 45° = s\ \cos\ 30°$, so substitute

$$t = \frac{\cos}{v_0 \cos\ 45°}s \qquad \text{and} \qquad x = s\cos\ 30°$$

in Eq. (*i*), this yields

$$x = \frac{v_0^2}{g}\left(1 - \frac{1}{\sqrt{3}}\right)$$

as before, and

$$s = \frac{x}{\cos\ 30°} = \frac{2}{3}(\sqrt{3} - 1)\frac{v_0^2}{g}$$

SP 4.10 At the ground $\tan\ \theta = v_y/v_x$, where $v_x = v_{0x} = v_0\cos\theta$.

$$v_y^2 = (v_0 \sin\theta)^2 + 2gh \text{ at } y = -\,h \text{ at ground}$$

Thus $$\tan\ \theta = \frac{\sqrt{(v_0 \sin\ \theta)^2 + 2\,gh}}{v_0 \cos\ \theta} = \left[\left(\frac{\sin\theta}{\cos\ \theta}\right)^2 - \frac{2\,gh}{(v_0 \cos\ \theta)^2}\right]^{1/2}$$

We need to minimize $\tan\ \theta$ with respect to variations in the launch angle θ. If $\tan\ \theta$ is a minimum, $\tan^2\theta$ is also a minimum. Minimizing $\tan^2\theta$ simplifies the math. Thus

$$\frac{d\,(\tan^2\theta)}{d\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta} - \frac{\sin^2\theta\,(-2\sin\theta)}{\cos^3\theta} - 2gh\frac{(-2)(-\sin\theta)}{\cos^3\theta} = 0$$

$$\frac{2\sin\ \theta}{\cos^3\theta}[\cos^2\ \theta + \sin^2\ \theta - 4gh\,] = 0 \quad \text{or} \quad \frac{\sin\ \theta}{\cos^3\ \theta}[1 - 4gh\,] = 0$$

This can only be true if $\sin\,\theta = 0$ or $\theta = 0°$. Substitute this back in $\tan\,\theta$:

$$\tan\,\theta = \frac{v_0 \sin^2\theta + 2\,gh}{v_0 \cos\,\theta} \qquad \sin 0° = 0 \qquad \cos\,0° = 1 \qquad \text{so } \theta = \tan^{-1}\frac{\sqrt{2\,gh}}{v_0}$$

SP 4.11 The time for the bullet to travel a range R is approximately

$$t = \frac{R}{v_x} \simeq \frac{R}{v_0} = \frac{200\,\text{m}}{900\,\text{m/s}} = 0.22\,\text{s}$$

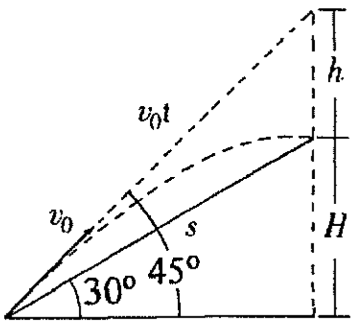
In this time it drops a distance of $h = 1/2gt^2 = 1/2\,(9.9\,\text{m/s}^2)(0.22\ \text{s})^2$.

(a) $h = 0.242\,\text{m}$.

(b) The sight is aligned on the assumption that after 200 m the bullet will have moved 0.242 m *perpendicular to the path.* But when shooting up a 45° slope, the bullet falls back at 45° to the trajectory not perpendicular to it. Thus as shown in the drawing, the bullet will strike above the target by an amount H, where

$$\frac{h}{h+H} = \sin\,45° = \frac{1}{\sqrt{2}}, \quad \text{so } H = (\sqrt{2}-1)h$$

Thus when shooting uphill, you must aim low to hit the target. The same reasoning applies when shooting downhill. *You must aim low when shooting either uphill or downhill.* Once my friend won a $50 bet with a gun nut who couldn't believe this.



A way of reasoning this out qualitatively is to consider the limiting cases of shooting straight up or straight down. Then gravity doesn't deviate the bullet from the barrel axis at all, and you will definitely hit high.

SP 4.12 The circumference of the earth is $s = 2\pi R$. The earth rotates once in time $T = 24$ h, so the velocity of a point on the equator is

$$v = \frac{2\pi R}{T} = \frac{(2\pi)(6370 \times 10^3\ \text{m})}{(24)(3600\ \text{s})} = 463\ \text{m/s}$$

The centripetal (radial) acceleration is

$$a_c = \frac{v^2}{R} = \frac{(463\,\text{m/s})}{(6.37 \times 10^6\,\text{m})} = 0.034\ \text{m/s}^2 \simeq 0.003g$$

SP 4.13 $v = \frac{2\pi r}{T}$ $\quad T =$ time for 1 rev $\quad T = \frac{1}{800}\text{min} = \frac{1}{800}(60\ \text{s}) = 0.075\ \text{s}$

$$v = \frac{2\pi\,(0.16\,\text{m})}{0.075\,\text{s}} = 13.4\,\text{m/s} \qquad a_c = \frac{v^2}{r} = \frac{(13.4\ \text{m/s})^2}{0.16\,\text{m}} = 1120\ \text{m/s}^2$$

SP 4.14 $$a_c = \frac{v^2}{r} = 0.05g$$

(a) If $v = 70\,\text{m/s}$, then:

$$r = \frac{(70\text{ m/s})^2}{(0.05)(9.8\text{ m/s}^2)} = 10\text{ km}$$

(b) If $r = 1.20$ km, then:

$$v = [(1200\text{ m})(0.05)(9.0\text{ m/s}^2)]^{1/2} = 24\text{ m/s}$$

SP 4.15 $a_T \simeq \frac{\Delta V_T}{\Delta t} = \frac{66\text{ m/s} - 60\text{ m/s}}{4\text{ s}} = 1.5\text{ m/s}^2$ $\qquad a_c = \frac{v^2}{r} = \frac{(63\text{ m/s})^2}{660\text{ m}} = 6.01\text{ m/s}^2$

$$a = \sqrt{a_T^2 + a_c^2} = \sqrt{(1.5\text{ m/s}^2)^2 + (6.0)\text{ m/s}^2} = 6.20\text{ m/s}^2$$

SP 4.16 Ground speed is

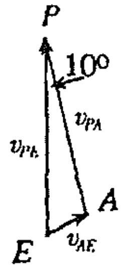
$$v_{PE} = \frac{1400\text{ km}}{1.9\text{ h}} = 737\text{ km/h}$$

The velocity of the plane with respect to air is 700 km/h directed 10° west of north, so the velocity vectors can be drawn. Using the law of cosines,

$$v_{AE}^2 = v_{PA}^2 + v_{PE}^2 - 2v_{PA}v_{PE}\cos$$

$$= (700)^2 + (737)^2 - 2(700)(737)\cos 10°$$

$$v_{AE} = 131\text{ km/h}$$

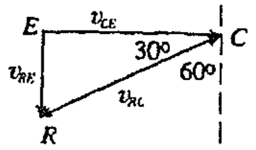


SP 4.17 Label car reference frame as C.

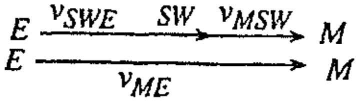
$$v_{CE} = 11\text{ m/s}$$

$$v_{RE} = v_{CE}\tan 30° = 11\tan 30° = 6.4\text{ m/s}$$

$$v_{ME} = v_{MSW} + v_{SWE} = 0.8\text{ m/s} + 1.20\text{ m/s} = 2.0\text{ m/s}$$



SP 4.18 $t = \frac{x}{v_{ME}} = \frac{80\text{ m}}{2\text{ m/s}} = 40\text{ s}$

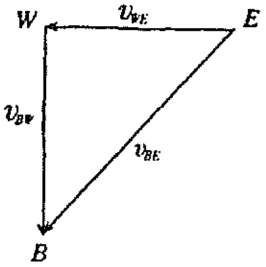


SP 4.19 $v_{WE} = 2.2\text{ m/s} \qquad v_{BW} = 4.8\text{ m/s}$

$$v_{BE} = \sqrt{v_{WE}^2 + v_{BW}^2} = 5.28\text{ m/s}$$

The speed perpendicular to the river is

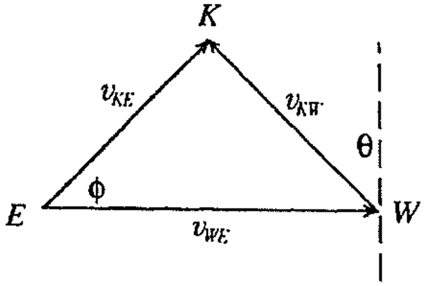
$$v_{BW},\ \text{so } t = \frac{d}{v_{BW}} = \frac{86\text{ m}}{4.8\text{ m/s}} = 17.9\text{ s}$$



In this time he drifts downstream a distance of

$$L = v_{WE}t = (2.2\text{ m/s})(17.9\text{ s}) = 39.4\text{ m}$$

SP 4.20 Suppose she heads upstream at an angle θ from straight across. Her velocity component headed across the river is $v_{KW} \cos\theta$. Thus the time to cross is $t = W/v_{KW}\cos\theta$. In this time she drifts downstream a distance $(v_{KE}/v_{KW}\cos\theta)W$.



W = river width

$v_{KW} = 0.90\,\text{m/s}$

$v_{WE} = 1.80\,\text{m/s}$

But $v_{KE}\cos\theta = v_{WE} - v_{KW}\sin\theta$, so

$$x = \left(\frac{v_{WE} - v_{KW}\sin\theta}{v_{KW}\cos\theta}\right)W$$

since

$$v_{WE} = 2v_{KW},\quad x = \left(\frac{2 - \sin\theta}{\cos\theta}\right)W \qquad (i)$$

Minimize x with respect to θ:

$$\frac{dx}{d\theta} = \left[\frac{-\cos\theta}{\cos\theta} + \frac{(2 - \sin\theta)(\sin\theta)}{\cos^2\theta}\right]W = 0$$

$$-1 + \frac{2\sin\theta - \sin^2\theta}{\cos^2\theta} = 0 \qquad -\cos^2\theta + 2\sin\theta - \sin^2\theta = 0 \qquad 2\sin\theta = 1 \qquad \theta = 30^\circ$$

From the drawing, I see that $v_{KE}\sin\theta = v_{KW}\cos\theta$ and $v_{KE}\cos\theta = v_{WE} - v_{KW}\sin\theta$. Divide

$$\frac{v_{KE}\sin\theta}{v_{KE}\cos\theta} = \tan\phi = \frac{v_{KW}\cos\theta}{v_{WE} - v_{KW}\sin\theta} = \frac{\sqrt{3/2}}{2 - 1/2} = \sqrt{3}$$

so $\phi = 60^\circ$. From Eq. (*i*),

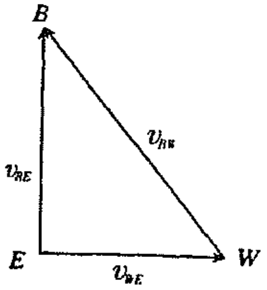
$$x = \frac{2 - \sin\theta}{\cos\theta}W \text{ so } 120 = \frac{2 - \sin 30^\circ}{\cos 30^\circ}W$$

Kate heads 30° upstream.

SP 4.21 The boat's speed with respect to earth is

$$v_{BE} = \frac{48\,\text{km}}{2.25\,\text{h}} = 21.3\,\text{km/h}$$

The water speed with respect to the earth is $v_{WE} = 23\,\text{km/h}$. Thus $v_{BW}^2 = (23\,\text{km/h})^2 + (21.3\,\text{km/h})^2$, and $v_{BW} = 31.3\,\text{km/h}$.



Chapter 5

Newton's Laws of Motion

I turn now to the question of dynamics, that is, the study of how forces influence motion. Dynamics is involved in understanding a wide range of phenomena. Begin by observing that forces result from interactions between two or more objects, so if A interacts with B, then B also interacts with A. Thus forces occur in pairs. The basic ideas needed are stated as Newton's laws of motion. It is helpful to consider the third law first.

5.1 NEWTON'S THIRD LAW OF MOTION

If body A pulls or pushes on body B, then body B also pulls or pushes on body A. The force on each body has the same magnitude, but the forces are oppositely directed.

The idea makes sense. Suppose Bill and Mary pull on opposite ends of a rope, as in a tug-of-war. Imagine that there is a spring scale in the rope, of the kind used in a market to weigh fish or vegetables. If Bill wants to know how hard he is pulling, he looks down and reads the scale. If Mary wants to know how hard she is pulling, she looks down and reads the same scale. Both persons will always see the same reading, and this force is called the **tension** in the rope. They pull with forces of equal magnitudes. The same is true if they are pushing. Suppose they are pushing on opposite sides of a bathroom scale. Each will read the same scale reading to determine how hard his or her push is. The readings must be the same for both people, since they read the same scale. This principle is true in *all* cases, no matter if the people are moving or if one is stronger or heavier. In SI units, force is measured in newtons, and $4.45\,\mathrm{N} = 1\,\mathrm{lb}$.

5.2 NEWTON'S FIRST LAW OF MOTION

Motion must be referred to a reference frame in order to be described. Reference frames in which Newton's laws are valid are called **inertial frames**. More specifically, an inertial reference frame is one in which Newton's first law of motion is valid.

Consider a body acted on by no net force. If it is at rest, it will remain at rest. If it is moving, it will continue to move with constant velocity.

By a "net force" I mean the vector sum of all external forces acting on the object.

For example, if you push on a book from the left with a horizontal force of 10 N and from the right with a force of 8 N, the net force acting is 2 N.

Any reference frame that moves with constant velocity with respect to an inertial reference frame is also an inertial reference frame. A reference frame that is accelerating with respect to an inertial reference frame will *not* be an inertial reference frame.

We will usually use a reference frame attached to the earth's surface (the "laboratory frame"), but this is not in fact an inertial frame, since the earth is rotating, and rotating objects are accelerating. However, for many purposes the rotation of the earth is sufficiently slow so that it will have a negligible effect on our calculations. However, for certain phenomena the effects of the earth's rotation are noticeable and must be included. Examples of the latter include motion of large air masses or of ocean currents, or the motion of an intercontinental ballistic missile or the trajectory of a long-range artillery shell.

5.3 NEWTON'S SECOND LAW OF MOTION

If a net force F acts on an object of mass m, the object will have acceleration a, where

$$\boxed{\mathbf{F} = m\mathbf{a}} \tag{5.1}$$

F is measured in newtons, **a** is measured in meters per second per second, and m is measured in kilograms.

Do not confuse mass with weight. Mass is proportional to weight at a given point, but as you move far away from the earth, where the force of gravity on an object, and hence its weight, decreases, the mass does not change. Mass is a measure of the amount of "stuff" in an object.

The force of gravity acting on an object is called the **weight** of the object. This force can be written as

$$\boxed{W = mg} \tag{5.2}$$

where W is the weight in newtons of an object of mass m (in kilograms), and g is a quantity that depends on the mass of the earth, the radius of the earth, and on a universal gravity constant. For motion of objects near the surface of the earth, g is almost constant, and I will take its value to be $g = 9.80\,\text{m/s}^2$. The quantity g is often called "the acceleration due to gravity." The reason for this term is the following: Suppose you release an object of mass m and allow it to fall freely under the influence of the gravity force. The net force acting on the object will be the gravity force, $-mg$. The force is negative because it is directed downward. Newton's second law becomes

$$ma = -mg, \quad \text{so } a = -g$$

Thus g is indeed the magnitude of the acceleration of a freely falling object. However, even when the object is not falling (perhaps it is at rest on a table), it is still acted on by the gravity force mg. Then no acceleration occurs, and it is somewhat misleading to refer to g as the "acceleration due to gravity." It is best just to call g by the name "gee." Whatever you do, don't call g "gravity."

It is common practice to "weigh" objects in grams or kilograms. This is incorrect, since weight is measured in newtons, not kilograms. However, since mass and weight are proportional, no great harm is done if you merely want to compare two things. If you double the mass, you will double the weight. However, in your calculations in physics be careful to distinguish between these two distinct concepts. Do not use mass m in kilograms where you should be using weight W in newtons.

We will encounter two classes of problems. When the **net force is not zero, acceleration will result**. Using our previous kinematic equations, we can then determine the motion. When the **net force acting is zero, no acceleration occurs**. This situation is called **equilibrium**. If an object, or collection of objects, remains at rest, it is obviously in equilibrium, and we can then deduce what forces are acting, if some of the forces are known. I consider these two classes of applications in the following section. In what follows, I will make some simplifying approximations, unless I indicate otherwise. I will neglect friction. I will neglect the variation in g with altitude. I will neglect the weight of ropes, assuming they are light compared to the other objects. I will treat objects as point masses. In so doing, I will not have to worry about rotations of objects of finite size. Later rotation will be taken into account in the study of the motion of extended rigid bodies. Finally, for the present I will consider problems where the forces all lie in a plane. This will not be a severe restriction, and it is not difficult to extend our treatment to forces in three dimensions. However, this is the situation for many interesting and practical problems, and the reduction in writing required helps make the concepts more clear.

5.4 APPLICATIONS OF NEWTON'S LAWS

In solving *all* problems involving forces, follow these procedures:

1. Draw a little picture, including sketches of people, cars, and so on, so that you are clear as to what is happening. Write down given information, and identify what is to be found.

2. Identify the forces that act *on* the object or system. These are called *external forces.* Do not include internal forces, for example, the forces between the atoms in the object. In our study of mechanics the only forces we will encounter are gravity, friction, normal forces exerted by surfaces, and tension (pulling forces, usually due to ropes). Show the forces in a **force diagram** (also called a **free-body diagram**). Slide all of the force vectors so that their tails are all at the one point that represents the object. Draw the force diagram with a straight-edge ruler, and do not make the drawing too small. It will usually occupy one-fourth page or more. Resolve the forces into components.

3. Determine the net force acting along each axis. Usually we use horizontal and vertical axes, but other perpendicular axes can be used if they are more convenient. For equilibrium problems, the net force will be zero. If the net force is not zero, use it to find the resulting acceleration, which can then be used to find velocity and displacement.

4. Use algebra to solve the equations obtained.

5. Substitute numerical values for the parameters in order to obtain a final answer.

5.4.1 EQUILIBRIUM PROBLEMS

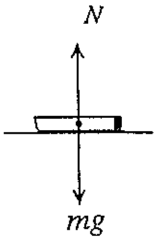
If the net external force acting on an object of a system is zero, the object is in **equilibrium**. Thus if $\mathbf{F} = m\mathbf{a} = 0$, then $\mathbf{a} = 0$. This means the object is at rest or else moving with constant velocity. Most of the problems we will encounter concern objects that are at rest, for example, a person standing still or a building or other structure. If several external forces act, say, $\mathbf{F}_1$, $\mathbf{F}_2$, $\cdots$, $\mathbf{F}_x$, then $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_x = \Sigma\mathbf{F}_{ix}$. This means $\Sigma F_{ix} = 0$ and $\Sigma F_{iy} = 0$. An easy way to apply these equations without having to worry about getting confused about the signs of the components F_{ix} and F_{iy} is simply to write

$$\boxed{F_{\text{left}} = F_{\text{right}} \quad \text{and} \quad F_{\text{up}} = F_{\text{down}}} \quad \text{in equilibrium} \qquad (5.3)$$

where F = total external force.

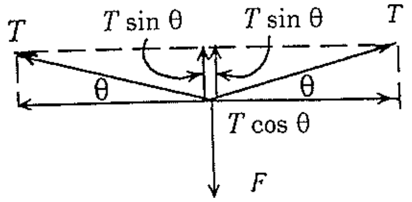
Problem 5.1 A book of mass 0.50 kg rests on a table. Draw the force diagram, and determine the upward force exerted by the table. Note: The force exerted by a surface perpendicular to the surface is called a **normal force**. In this context the word normal means "perpendicular." I will label such forces by the letter N.

Solution The forces acting on the book are the force of gravity mg downward and the upward normal force exerted by the table. Since the book is in equilibrium, $F_{\text{up}} = F_{\text{down}}$ or $N = mg$. Thus $N = (0.50\,\text{kg})(9.8\,\text{m/s}^2) = 4.9$ N. (Note: Remember that $1\ \text{kg}\cdot\text{m/s}^2 = 1$ N.) Caution: Do not confuse weight mg in newtons with mass m in kilograms.



Problem 5.2 Once while elk hunting with a couple of mountain men in Idaho, our pickup truck got stuck in the mud. My compatriots got it out by using the following trick. They tied a steel cable tautly between the truck and a nearby tree in front of the truck. Then they pulled sideways with force F on the midpoint of the cable. Sure enough, the truck popped out of the mudhole. For such an arrangement, if the force F is 400 N (about 90 lbs), what force does the cable exert on the truck if the angle θ in the drawing was 10°?

Solution Let T = tension in the cable = force on the truck. In a cable the tension is the same everywhere. Just before the truck moves, it is in equilibrium, so in the force diagram here,



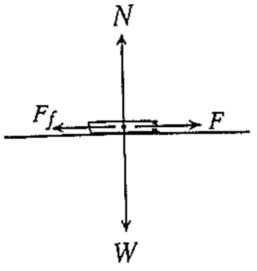
$F_L = F_R$ or $T \cos \theta = T \cos \theta$

and $F_U = F_D$ or $\sin \theta + T \sin \theta = F$

So $T = \frac{F}{2 \sin \theta} = 1150 \text{ N}$

Thus pulling sideways has multiplied the force on the truck by a factor of almost 3 compared to what could have been obtained by pulling straight ahead. You might wonder where the "extra" force came from. The answer is that the tree is pulling on the cable, and this pull is also exerted on the truck For this method to work, you need to use a steel cable that won't stretch, and it must be taut in order to minimize the angle θ.

Friction forces are encountered in many mechanical problems. When two surfaces are in contact, each can exert a force (called *friction*) on the other that is parallel to the surface. This force depends on the roughness of the surfaces. Friction forces are a little tricky in that they are *reactive forces.* By this I mean that they push back in response to another applied force. For example, if a book rests on a level table and you don't touch it, the friction force acting on it is zero. If you push lightly on the book with a force of $0.1N$, the friction force will push back with a force of $0.1N$, and the book won't move. If you increase your force to $0.2N$, the friction force also increases to $0.2N$, and still the book doesn't move. If you keep increasing your force, you will finally reach a point where the book does begin to move. The maximum friction force available has then been surpassed. The maximum friction force that can be exerted on the book depends on the nature of the book surface and the table surface and also on how strongly the two are pressed together, that is, on the normal force exerted by the table on the book. Thus we write the maximum friction force as



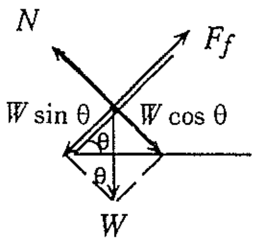
$$\boxed{F_f = \mu N} \tag{5.4}$$

where μ = coefficient of friction, a dimensionless number.

The coefficient of friction depends on the nature of the two surfaces. Typically μ is greater when the objects are stationary than when one object is sliding over the other, because sliding tends to break off the little sharp points (on a microscopic scale) sticking out of any surface. The coefficient of static friction is greater than the coefficient of kinetic (or sliding) friction.

Problem 5.3 The coefficient of friction between a load of sand and the bed of the dump truck in which it is carried is 1.10. At what angle to horizontal does the truck bed have to be tilted before the sand starts to slide out?

Solution The force diagram is drawn here. It is easiest here to resolve the forces along axes parallel and perpendicular to the surface of the bed because the normal force and the friction force are already along these directions. Thus we have to resolve only the weight mg. It is not incorrect to use the x and y axes, but so doing requires more algebra.



Just before the sand slides, it is in equilibrium, so

$$F_{\text{into the bed}} = F_{\text{out of the bed}} \qquad \text{and} \qquad F_{\text{down the slope}} = F_{\text{up the slope}}$$

Thus $$W\cos\theta = N \qquad W\sin\theta = F_f = \mu N$$

Divide these equations:

$$\frac{W\sin\theta}{W\cos\theta} = \frac{\mu N}{N} \qquad \text{so } \tan\theta = \mu = 1.1,\ \theta = 48°$$

Incidentally, the maximum angle for which no sliding occurs is called the "angle of repose." Read the fascinating novel by this name by Wallace Stegner. It is a great book, and Stegner gets all of his physics metaphors exactly right.

Problem 5.4 A mechanic tries to remove an engine from a car by attaching a chain to it from a point directly overhead and then pulling sideways with a horizontal force F. If the engine has mass 180 kg, what is the tension in the chain when it makes an angle of 15° with vertical? What is the force F?

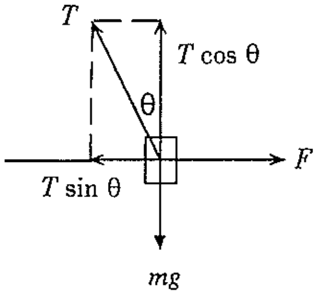
Solution Resolve forces, with $F_{\text{up}} = F_{\text{down}}$ and $F_L = F_R$:

$$T\cos\theta = mg \qquad T\sin\theta = F$$

Divide $$\frac{\sin\theta}{\cos\theta} = \frac{F}{mg} \qquad F = mg\tan\theta$$

$$F = (180\,\text{kg})(9.8\,\text{m/s}^2)\tan 15° = 473\,\text{N}$$

$$T = \frac{F}{\sin\theta} = \frac{473}{\sin 15°} = 1830\,\text{N}$$

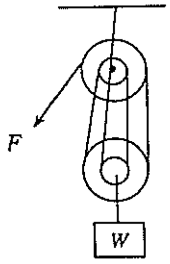


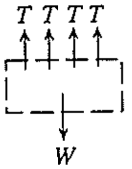
Problem 5.5 A block and tackle is a simple machine used to lift heavy weights. For the arrangement shown here, what force F must be exerted to lift a load of weight W?

Solution Let the "system" be the load and the lower pulley (of negligible weight). The force diagram is as shown. The tension in the rope is everywhere $T = F$.

$$F_{\text{up}} = F_{\text{down}} \text{ so } 4T = W, \quad \text{or } T = F = \frac{W}{4}$$

The ratio of the weight lifted to the force applied is called the **mechanical advantage** of the machine. Here the MA = 4.





Problem 5.6 A girl moves her brother on a sled at a constant velocity by exerting a force F. The coefficient of friction between the sled and the ground is 0.05. The sled and rider have a mass of 20 kg. What force is required if (a) she pushes on the sled at an angle of 30° below horizontal? (b) She pulls the sled at an angle of 30° above horizontal?

Solution (a) The sled is in equilibrium since the velocity is constant. Thus $F_{\text{up}} = F_{\text{down}}$ and $F_L = F_R$.

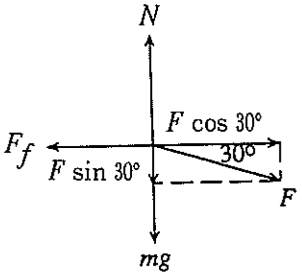
$$N = mg + F \sin 30°$$

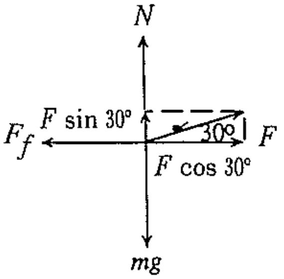
$$F_f = \mu N = F \cos 30°$$

so $$\mu\,(mg + F \sin 30°) = F \cos 30°$$

$$F = \frac{\mu mg}{\cos 30° - \mu \sin 30°}$$

$$F = \frac{(0.05)(20\text{ kg})(9.8\text{ m/s}^2)}{\cos\ 30° - 0.05\ \sin 30\ °} = 11.7\text{ N}$$





(b) $$N + F \sin 30° = mg$$

$$F_f = \mu N = F \cos\ 30° \qquad \mu\,(mg - F \sin\ 30°) = F \cos\ 30°$$

$$F = \frac{\mu mg}{\cos\ 30° - \mu \sin\ 30°} = \frac{(0.05)(20\text{ kg})(9.8\text{ m/s}^2)}{\cos\ 30° + 0.05\ \sin\ 30°} = 11.0\text{ N}$$

The force required in (b) is less than for (a) because in (b) the force F angles up and supports some of the weight. This reduces N and hence F_f.

5.4.2 NONEQUILIBRIUM PROBLEMS

When a net force acts on a system, the system will have an acceleration given by $\mathbf{F} = m\mathbf{a}$. If we know $\mathbf{F}$, we can find the acceleration $\mathbf{a}$, and knowing $\mathbf{a}$ and the initial conditions, we can use the kinematic equations, Eq. 3.10, to find displacement and velocity.

Problem 5.7 A woman is wearing her seat belt while driving 60 km/h. She finds it necessary to slam on her brakes, and she slows uniformly to a stop in 1.60 s. What is the average force exerted on her by the seat belt (neglecting friction with the seat)? Express the result as a multiple of the woman's weight.

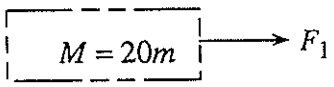
Solution If she slows from v_1 to v_2 in time t, her average acceleration is

$$a = \frac{v_2 - v_1}{t} = \frac{0 - 60}{1.60\,\text{s}} = -10.4\ \text{m/s}^2$$

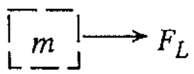
Thus the average force exerted on her is $F = ma$. Her weight is $W = mg$, so

$$m = \frac{W}{g} \qquad \text{and} \qquad F = \frac{Wa}{g} = \frac{10.4}{9.8}W = 1.06W$$

Problem 5.8 A locomotive pulls 20 boxcars, each with mass of 56,000 kg. The train accelerates forward with acceleration 0.05 m/s^2. (a) What is the force exerted by the coupling between the locomotive and the first car? (b) What is the force exerted by the coupling between the last car and the next to last car?



Solution (a) View the 20 cars as the system. Then $F_1 = 20ma$, where $m = 56{,}000$ kg. Thus $F_1 = (20)(56{,}000\ \text{kg})(0.05\ \text{m/s}^2) = 56{,}000$ N. (b) View the last car as the system $F_L = ma = (56{,}000\ \text{kg})(0.05\ \text{m/s}^2) = 2300$ N.



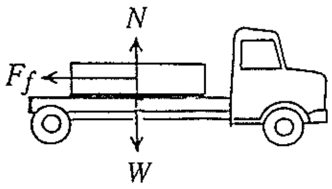
Problem 5.9 A woman (mass 50 kg) and her son (mass 25 kg) face each other on ice skates. Placing the palms of their hands together, they push each other apart, with the mother exerting an average force of 40 N on her son. What will be the acceleration of each during this process?

Solution According to Newton's third law, each person will experience a force of the same magnitude, 40 N. Thus the accelerations will be

$$a_m = \frac{F}{m_m} = \frac{40\,\text{N}}{50\,\text{kg}} = 0.80\,\text{m/s}^2 \qquad a_s = \frac{F}{m_s} = \frac{40\,\text{N}}{25\,\text{kg}} = 1.60\,\text{m/s}^2$$

Problem 5.10 Once during the Great Depression of 1933 my father found temporary work driving a big flat bed truck loaded with steel. While going down a hill in Los Angeles, the brakes went out. This situation was a little stressful. If he hit something, either he would be injured in the crash, or else the steel would slide forward and wipe out the cab and driver. Fortunately an alert motorcycle cop saved the day by clearing traffic for a run-out. Suppose a load of steel is held in place only by friction, with a coefficient of friction of 0.4. What is the shortest stopping distance on level ground when moving 20 m/s (about 45 mi/h) if the load is not to slide forward into the cab?

Solution The force diagram for the load is as shown here. The only horizontal force is the force of friction, and this is the net force acting on the load.



Thus

$$F = ma = -F_f$$

$$N = mg \quad \text{and} \quad F_f = \mu N$$

So $\quad a = -\frac{\mu mg}{m} = -\mu g$

From Eq. 3.10, $v^2 = v_0^2 + 2ax = 0$ when stopped. Thus

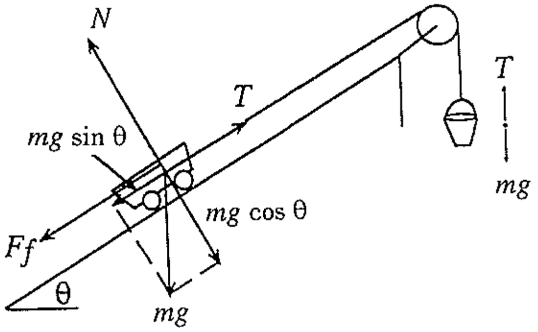
$$x = \frac{v_0^2}{2a} = \frac{v_0^2}{2\mu g}$$

Problem 5.11 Here is a famous classic problem that will make you think. A rope is passed over a pulley suspended from a tree branch, and a stalk of bananas is tied to one end. A monkey hangs from the other end of the rope, and the bananas and the monkey are balanced. Now the monkey starts climbing up the rope. What will happen to the bananas? Will they stay in the same place, or will they move up away from the ground, or will they move down toward the ground?

Solution Look at the force diagram for the monkey. His weight mg acts downward, and the rope tension T acts upward. If the monkey is to start moving up from rest, he must accelerate upward, which means there must be a net upward force acting on him. The net upward force on the monkey is $T - mg$, where T is the tension in the rope. But the tension is the same everywhere in a rope, so the tension at the end of the rope attached to the bananas is also T, greater than mg. Thus the bananas experience the same upward force as does the monkey, and so the bananas will move up with the same acceleration and velocity as the monkey. Both will move higher from the ground at the same rate. The net upward force on the system made up of the bananas plus the monkey is provided by the pulley.

Problem 5.12 Once my Boy Scout troop tried to improvise a scheme to pull an old ore car up out of a sloping mine shaft. The idea is illustrated here. We were going to divert a stream so that water ran into a bucket attached to the ore car. When enough water filled the car, it was supposed to move up the track. If the ore car had mass of 80 kg, and the track was inclined at 15° above horizontal, (a) what mass of water would be required to start the car moving if friction was negligible? (b) If the friction coefficient between the car and the track was 0.20, what mass of water would be needed to start the car moving? (c) With friction present, suppose water is added until the car is just about to move. Now an additional 4 kg of water is added to the bucket with the wheels of the car locked. When the wheels are unlocked, how long will it take the car to move 34 m up the track?

Solution a) Draw the force diagram. Resolve the forces into components parallel to the track and perpendicular to the track. In equilibrium, so for the car, $mg\sin\,\theta = T$ and for the bucket, $T = Mg$. Thus $Mg = mg\sin\theta$ and $M = m\sin\theta = 80\sin 15° = 20.7$ kg.



(b) With friction present, in equilibrium, $T = mg\sin\,\theta + F_f = mg\ \sin\,\theta + \mu N$. $\mu N = mg\ \cos\,\theta$, so $T = mg\ \sin\,\theta + \mu\, mg\ \cos\,\theta$. For the bucket, $T = Mg$, so $Mg = mg\ (\sin\theta + \mu\ \cos\,\theta)$. $M = m\ (\sin\,\theta + \mu\ \cos\,\theta) = (80\,\text{kg})(\sin 15° + 0.20\cos 15°) = 36.2$ kg.

(c) If $\Delta m = 4\,\text{kg}$ is added to the bucket, the net force will then be $F = \Delta mg$. Thus $F = \Delta mg = (m + M + \Delta m)a$.

$$a = \frac{\Delta m}{m + M + \Delta m}g = \frac{4}{80 + 36.2 + 4}(9.8\,\text{m/s}^2) = 0.33\,\text{m/s}^2$$

From Eq. 3.10, $x = v_0 t + \frac{1}{2}\,at^2$, where $v_0 = 0$, so

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{(2)134\text{ m}}{0.33\,\text{m/s}^2}} = 14.4\,\text{s}$$

Problem 5.13 A person whose weight is 600 N stands on a bathroom scale in an elevator. What will the scale read when the elevator is (a) moving up or down at constant speed? (b) Accelerating up with acceleration $0.5g$? (c) Accelerating downward with acceleration $0.5g$? (d) Accelerating downward with acceleration g?

Solution The force diagram includes two forces: the gravity force mg downward and the normal force N, exerted upward by the surface of the scale. This normal force is the scale reading.

(a) Constant velocity means $a = 0$, so equilibrium and $N = W = 600\,\text{N}$.
(b) $N - W = ma = +\,0.5mg = 0.5W$, so $N = W + 0.5W = 1.5W$.
(c) $N - W = ma = -0.5\,mg = -0.5W$, so $N = 0.5W$.
(d) $N - W = ma = -mg = -W$, so $N = 0$.

The last case, when the scale reading is zero, represents what is called "effective weightlessness." The elevator is falling with acceleration $-g$, as is the person. Thus the person does not press down on the elevator. He is seemingly "weightless." This is the situation with the astronauts in an orbiting space vehicle. The vehicle and everything in it are falling freely, and hence they all seem weightless. You have probably seen pictures where the astronauts, their pencils, and their sandwiches and other loose equipment float weightlessly around the spaceship. Everything is falling toward the earth with acceleration g. Because they are also moving sideways, they do not actually get closer to the earth as they fall.

Problem 5.14 A tire manufacturer performs road tests that show that a new design of tire has an effective coefficient of friction of 0.83 on a dry asphalt roadway. Under these conditions, what would be the stopping distance for a car traveling 50 km/h (about 31 mi/h? 100 km/h (about 62 mi/h)?

Solution The net force acting on the car is the force of friction $F = \mu N$. Vertical forces are balanced, so $N = mg$. Thus

$$a = \frac{-F}{m} = -\frac{\mu mg}{m} = -\mu g \qquad v^2 = v_0^2 + 2\,ax = 0$$

when stopped, so

$$v_0^2 - 2\mu g\,x = 0 \qquad x = \frac{v_0^2}{2\mu g}$$

At 50 km/h,

$$x = \frac{\left[(50)\left(\frac{1000\,\text{m}}{3600\,\text{s}}\right)\right]^2}{(2)(0.83)(9.8\ \text{m/s}^2)} = 11.9\,\text{m}$$

At 100 km/h,

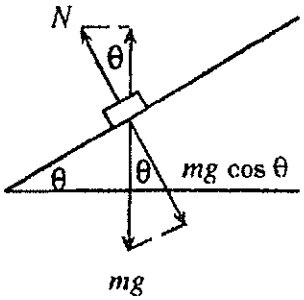
$$x = 47.4\,\text{m}$$

Notice that the stopping distance varies as the *square* of the speed. Thus, doubling your speed increases your stopping distance by a factor of 4. SPEED KILLS!

Problem 5.15 Consider a mass with initial velocity v_0. It can be launched as a projectile with its initial velocity elevated at angle θ above horizontal, or it can be launched up a frictionless plane inclined at angle θ above horizontal. In which case will the mass reach the greatest elevation, or will the maximum elevation be the same in each case? To answer this, calculate the maximum elevation reached in each case.

Solution For the projectile, $v_y^2 = v_{0y}^2 - 2gy_p$ from Eq. 3.10, $v_{0y} = v_0 \sin\theta$ and $v_y = 0$ at the highest point. Thus

$$y_p = \frac{v_0^2 \sin^2\theta}{2g}$$



For the particle sliding up the plane, draw the force diagram.

$$F_{\text{into plane}} = F_{\text{out of plane}} \qquad mg\cos\theta = N$$

Thus the net vertical force is

$$F_y = -mg + N\cos\theta = -mg + mg\cos^2\theta$$

Since $F_y = ma_y$,

$$a_y = \frac{F_y}{m} = \frac{-mg + mg\cos^2\theta}{m} = -g + g\cos^2\theta$$

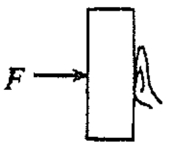
$$v_y^2 = v_{0y}^2 + 2a_y y = 0$$

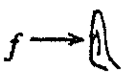
at the highest point, so

$$y = -\frac{(v_0\sin\theta)^2}{(2)(-g + g\cos^2\theta)} = \frac{v_0^2\sin^2\theta}{2g\,(1-\cos^2\theta)}$$

Thus $y > y_p$, and the sliding mass rises higher.

Problem 5.16 In an interesting lecture demonstration, I sometimes show the property of inertia as follows: I hold a lead brick (mass about 9 kg) in my hand and give it a mighty blow with a big hammer. I can hardly feel the force exerted on my hand. Were the hammer to strike my hand directly, it would surely be broken to smithereens. In the construction trades this trick of "backing" something with a massive object to reduce the force exerted on the "backer" is often used. To understand what is happening, approximate your hand as an isolated mass m in contact with a larger mass M (the brick). A force F is applied to the brick. Calculate the force f transmitted to your hand. What is f if $F = 100N$, $M = 9$ kg, $m = 1$ kg?

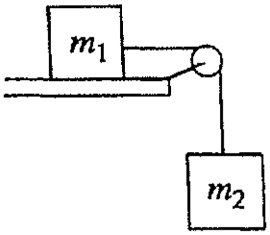




Solution First consider the block plus the hand as the "system" subject to force F. Then $F = (M + m)a$, $a = F/(M + m)$. Now consider the hand alone, subject to force f: $f = ma$ where $a = F/(M + m)$, since hand and block move together. Thus

$$f = \frac{m}{M+m}F \qquad f = \frac{1}{10+1}(100) = 9N$$

Problem 5.17 Blocks of masses m_1 and m_2 are connected by a light string. The coefficient of friction between m_1 and the table surface is μ. Determine the acceleration of the blocks and the tension in the string if $m_1 = 4$ kg, $m_2 = 3$ kg, and $\mu = 0.30$.



Solution Draw the force diagram for each block. For m_1,

$$N = m_1 g \qquad F_f = \mu N = \mu m_1 g$$

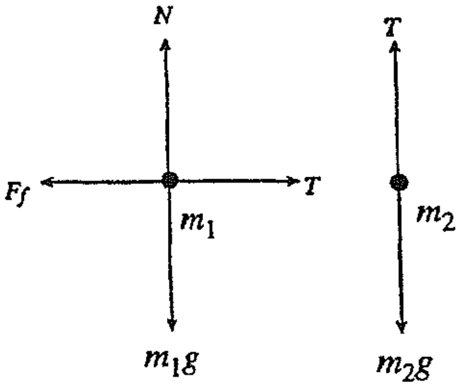
$$F_{\text{net}} = T - F_f = T - \mu m_1 g = m_1 a$$

For m_2, $$m_2 g - T = m_2 a$$

Solve $$T = m_2 a - m_2 g$$

so $$m_2 g - m_2 a - \mu m_1 g = m_1 a$$

$$a = \frac{m_2 - \mu m_1}{m_1 + m_2} g \qquad T = \frac{m_1 m_2}{m_1 + m_2}(1 + \mu) g$$



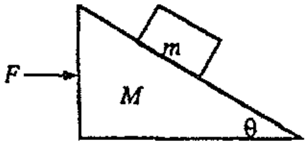
Substitute numbers: $T = 21.8N$, $a = 2.52\,\text{m/s}^2$.

Observe that we could also solve this problem by first considering both blocks together to form the system. In this case we would have

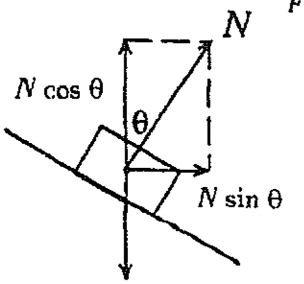
$$m_2 g - F = (m_1 + m_2) a$$

where $F = \mu N = \mu m_1 g$. This yields the acceleration immediately. We can then obtain the tension from $m_2 g - T = m_2 a$.

Problem 5.18 A small block of mass m is placed on a wedge of angle θ and mass M. Friction is negligible. What horizontal force must be applied to the wedge so that the small block does not slide up or down the wedge surface?



Solution First consider both blocks together: $F = (m + M)a$. Now look at the small block alone. The surface exerts a normal force on it. It does not move vertically, so $N\cos\theta = mg$. Horizontally, $N\sin\theta = ma$. Thus



$$\left(\frac{mg}{\cos\theta}\right)\sin\theta = m\left(\frac{F}{m+M}\right)$$

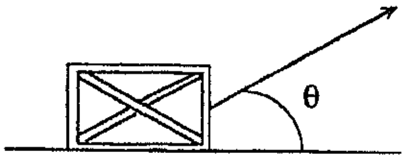
$$F = (m+M)g\tan\theta$$

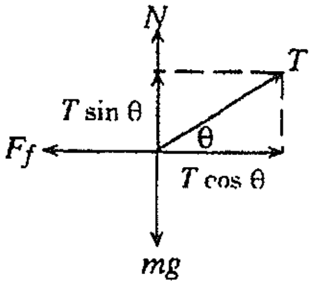
Problem 5.19 A packing crate of mass m is pulled across the floor at constant velocity by means of a cable attached to the front of the crate. The cable makes an angle θ with the floor. The coefficient of friction between the floor and the crate is μ. What value of θ will make the tension a minimum?

Solution: In equilibrium, so $F_{\text{up}} = F_{\text{down}}$ and $F_L = F_R$.

$$N + T\sin\theta = mg$$

$$F_f = \mu N = T\cos\theta$$





Solve for T:

$$N = mg - T\sin\theta$$

$$\mu(mg - T\sin\theta) = T\cos\theta \qquad T = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

Minimize T with respect to variations in θ by requiring that

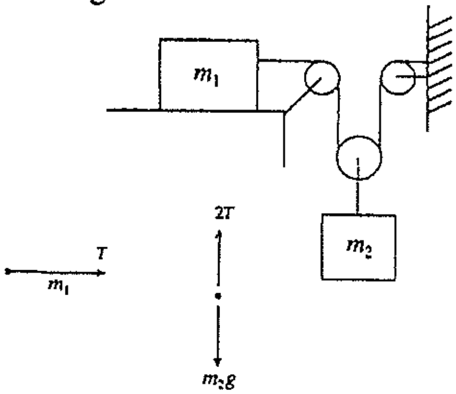
$$\frac{dT}{d\theta} = 0, \qquad \text{so } \frac{dT}{d\theta} = \mu mg\left[-\left(\frac{1}{\cos\theta + \mu\sin\theta}\right)^2(-\sin\theta + \mu\sin\theta)\right] = 0$$

Thus $\frac{dT}{d\theta} = 0$ if $-\sin\theta + \mu\cos\theta = 0$, or $\mu = \tan\theta$

Problem 5.20 Two masses are connected as shown here. Friction is negligible. What is the acceleration of each mass? What is the tension in the string?

Solution Study the drawing carefully and you will see that when m_1 moves 2 cm, m_2 drops only 1 cm. Thus, $a_1 = 2a_2$. For $m_1, T = m_1a_1$. For m_2, $m_2a_2 = m_2g - T$. Solve $m_2a_2 = m_2g - m_1a_1 = m_2g - 2m_1a_2$:

$$a_2 = \frac{m_2}{2m_1 + m_2}g \quad a_1 = \frac{2m_2}{2m + m_2}g \quad T = \frac{2m_1m_2}{2m_1 + m_2}g$$



Problem 5.21 On a level road the stopping distance for a certain car going 80 km/h is 32 m. What would be the stopping distance for this car when going downhill on a 1:10

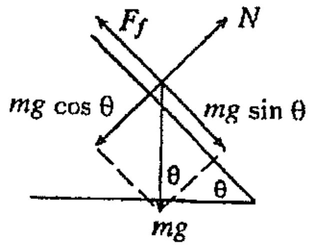
grade? (A grade of 1:10 means the elevation drops 1 m for a forward travel of 10 m along the roadway.)

Solution $v_0 = 80\,\text{km/h} = 80(\frac{1000\,\text{m}}{3600\,\text{s}}) = 22.2\,\text{m/s}$ $\theta = \sin^{-1}\frac{1}{10} = 5.74°$

On level ground, $-F_f = -\mu N = -\mu mg = ma$. So $a = -\mu g$ and

$$v_2 = v_0^2 + 2\,ax = 0 \qquad x = \frac{-v_0^2}{-2\mu g} \qquad \mu = \frac{v_0^2}{2\,gx} = \frac{(22.2\,\text{m/s})^2}{(2)(9.8\,\text{m/s}^2)(32\,\text{m})} = 0.79$$

Let a_s = the acceleration along the slope: $F_s = mg\sin\theta - F_f = ma_s$. Thus $ma_s = mg\sin\theta - \mu mg\cos\theta$. The stopping distance on the slope is given by $v^2 = v_0^2 + 2\,a_s s = 0$ when stopped, where $a_s < 0$. Thus



$$s = -\frac{v_0^2}{2\,a_s} = \frac{v_0^2}{2(g\sin\theta - \mu g\cos\theta)} = -\frac{(22.2\,\text{m/s})^2}{2(9.8\,\text{m/s}^2)[(\sin 5.7°\,(0.79)\,\cos 5.7°)]}$$

$$s = 36.8\,\text{m} \qquad \text{stopping distance on slope}$$

Problem 5.22 An object falling through the air at high speed experiences a drag force which can be expressed approximately as

$$F_D = \tfrac{1}{2}\rho_A A C_D v^2$$

Here ρ_A is the density of air, C_D is a drag coefficient that depends on the shape and texture of the falling object, and A is the projected area of the object as seen looking up from the ground. C_D is a dimensionless number between 0 and 1. (a) Determine the maximum speed (called the **terminal velocity**) a falling object reaches in the presence of this drag force. (b) How does the terminal velocity of an object depend on its size? To answer this, calculate the ratio of the terminal velocities for two spherical hailstones, one of radius r_1 and a larger one of radius r_2.

Solution (a) As the object falls, v gets larger and larger, until finally $F = 1/2\ \rho_A AC_D v^2 - mg = ma = 0$. When $a = 0$, v = constant, where $1/2\ \rho_A AC_D\ v_T^2\ - mg = 0$. Thus

$$v_T = \sqrt{\frac{2mg}{\rho_A AC_D}}$$

(b) For a sphere of ice, $m =$ (density)(volume) $= 4/3\ \pi R^3\rho$. The projected area seen from below is $A = \pi R^2$. Thus

$$v_T = \sqrt{\frac{2mg}{\rho_A A C_D}} = \sqrt{\frac{(2)(\rho)(4\pi R^3)g}{\rho_A \pi R^2 C_D}} = \sqrt{\frac{8\rho g}{\rho_A C_D}}\sqrt{R}$$

$$v_T = \kappa\sqrt{R}$$

where

$$\kappa = \sqrt{\frac{8\rho g}{\rho_A C_D}}$$

For two ice spheres,

$$\frac{v_{T_2}}{v_{T_1}} = \frac{\kappa\sqrt{r_2}}{\kappa\sqrt{r_1}} = \sqrt{\frac{r_2}{r_1}}$$

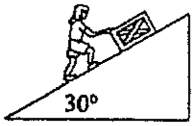
Thus large objects fall faster than small ones of the same shape. Big hailstones can flatten a wheat field, whereas small raindrops don't hurt it. Typical terminal velocities are as follows: 14.5 mi/h for a raindrop, 105 mi/h for a bullet, 140 mi/h for a person, and 145 mi/h for a 1000-lb bomb.

5.5 SUMMARY OF KEY EQUATIONS

Newton's third law:	If A exerts force $\mathbf{F}_{AB}$ on B and B exerts force $\mathbf{F}_{BA}$ on A, then $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$.
Newton's second law:	$\mathbf{F} = m\mathbf{a}$.
Newton's first law:	If the net external force $\mathbf{F} = 0$, then $\mathbf{a} = 0$ and $\mathbf{v} =$ constant
Equilibrium:	If $\mathbf{F} = 0$, then $\mathbf{a} = 0$, and $\mathbf{F}_{\text{up}} = \mathbf{F}_{\text{down}}$, and $\mathbf{F}_{\text{left}} = \mathbf{F}_{\text{right}}$.

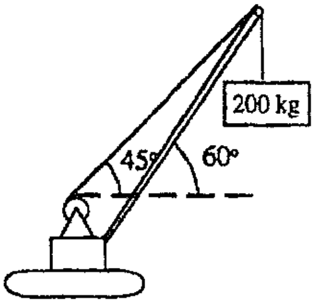
Supplementary Problems

SP 5.1 In an attempt to keep a packing box of mass m from sliding down a ramp inclined at angle 30° above horizontal, a woman exerts a horizontal force F. Assuming friction is negligible, what minimum force must she exert?



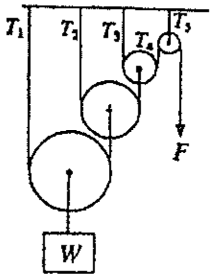
SP 5.2 In a daredevil rescue attempt, a marine holds on to the landing gear of a hovering helicopter with one hand while with his other hand he reaches down to lift a buddy below him. If the upper marine has a weight of 450 N and the lower marine weighs 350 N, what force is exerted by the upper marine's upper arm? By his lower arm?

SP 5.3 A crane lifts a mass of 200 kg with the arrangement shown here. Determine the force exerted by the boom and the tension in the cable.

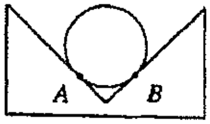


SP 5.4 A uniform heavy cable of mass m is attached to two eyebolts on a ceiling. The line tangent to the cable makes an angle of 30° with the ceiling at each end of the cable. Determine the force exerted on each eyebolt and the tension at the midpoint of the cable.

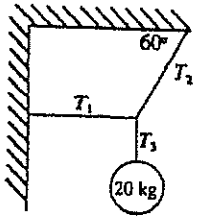
SP 5.5 What force F must be exerted on the block and tackle system shown here if the weight W is stationary? Determine the tensions T_1, T_2, T_3, T_4, and T_5. The pulleys have negligible weight.



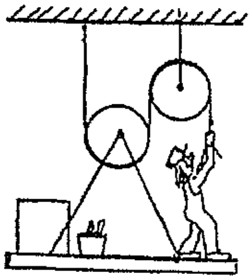
SP 5.6 A steel ball bearing of mass 0.020 kg rests in a 90° groove in a track. What force does it exert on the track at point A and at point B?



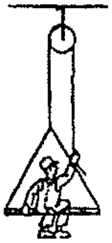
SP 5.7 Find the tension in each cord for the 20-kg mass shown suspended here.



SP 5.8 A painter who weighs 600 N stands on a platform, as shown here. The platform, paint, brushes, and so on weigh 400 N. What is the tension in the rope the painter is holding when the platform is motionless?

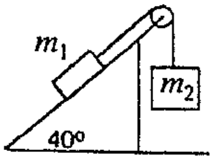


SP 5.9 A painter of mass 80 kg sits in a bosun's chair of mass 10 kg. He pulls on the rope he is holding in order to accelerate himself up. In so doing, he presses down on the seat with a force of 392 N. (a) What is his acceleration? (b) What is the tension in the rope supporting the pulley?



SP 5.10 You can make a simple accelerometer with which to measure approximately the acceleration of your Ferrari. Tie a small weight to the end of a string and let it hang vertically as a pendulum. When holding this instrument in your car while it is accelerating, the string will deviate from vertical by an angle θ. Derive an expression for the acceleration of the car as a function of θ. (You will have to make a protractor with which to measure the angle θ.)

SP 5.11 Mass m_1 slides without friction on a plane inclined at 40° above horizontal. It is attached to a second mass m_2 by a light string. If $m_1 = 5$ kg and $m_2 = 4$ kg, determine the acceleration of each block and the tension in the string.



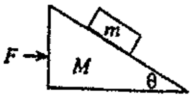
SP 5.12 I always worry about being trapped on a high floor of a hotel when it catches fire. How will I escape? I have a plan worked out. I'll tie together my sheets and drapes and whatever else I can find and make a rope, which I will then slide down. Unfortunately, I've found that typically such a rope will support only about three-fourths of my weight. However, if I slow my descent with friction by gripping the rope, I might make it. (a) At what maximum acceleration can I descend without breaking the rope? (b) At what speed will I hit the ground 20 m below in my slowest descent? (c) At what speed will I hit the ground if I just jump for it with no rope?

SP 5.13 A flea is a remarkably animal. It can leap to a height of about 32 cm (about 200 times its body length) when taking off at an angle of 60° above horizontal. Assuming a flea mass of 5.0×10^{-7} kg and a push-off time of 10^{-3}s, calculate (a) the average force exerted on the floor, expressed as a multiple of the flea's weight, and (b) the average acceleration of the flea during lift-off.

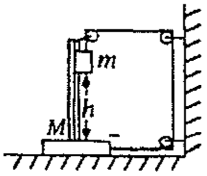
SP 5.14 Some amateur hot air balloonists find themselves accelerating downward with acceleration a at a moment when the mass of the balloon plus the balloonists is M. They want to accelerate upward at this same rate, so they throw out ballast of mass m. Determine m.

SP 5.15 When a small object such as a blood cell or a macromolecule or a silt particle falls through a viscous fluid under the influence of gravity, it experiences a drag force of the form bv, as long as v is small. Under these circumstances the particle achieves a steady velocity called the **sedimentation velocity.** Measurement of the sedimentation velocity enables us to learn something about the falling particle. Calculate the sedimentation velocity in terms of b, the particle mass, and g.

SP 5.16 A small block of mass m can slide without friction on a wedge of mass M inclined at angle θ. What horizontal force F must be applied to the wedge if the small block is not to move with respect to the wedge?

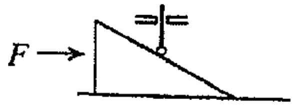


SP 5.17 A principle used in certain interlock systems is illustrated here. Mass m slides on a vertical track attached to a base unit of mass M. The two masses are joined by a light cord, as shown. Mass m is



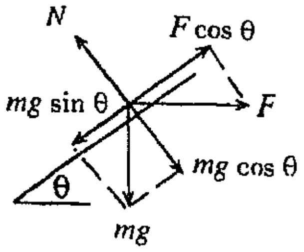
released from rest. Determine how long it takes to fall a distance h to the base below.

SP 5.18 One type of cam arrangement operates on the principle illustrated here. A follower rod of mass 0.300 kg slides vertically in a lubricated bushing. At its end is a bearing that rests on a movable wedge of angle 15° and mass 0.150 kg. The wedge can move horizontally. Friction is negligible throughout the device. What horizontal force must be applied to the wedge to impart an upward acceleration of 2.0g to the follower rod?



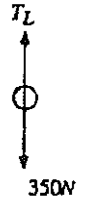
Solutions to Supplementary Problems

SP 5.1 $F\cos\theta = mg\sin\theta$ so $F = mg\tan\theta$.



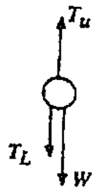
SP 5.2 Lower arm:

$T_l = 350\text{ N}$



Upper arm:

$$T_u = T_l + W$$
$$= 350\text{ N} + 450\text{ N}$$
$$= 800\text{ N}$$



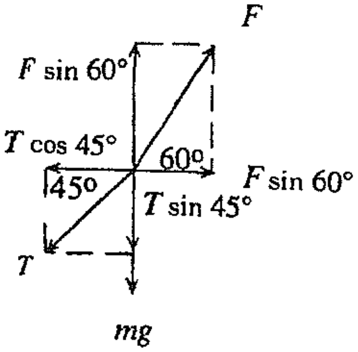
SP 5.3 $F_{\text{up}} = F_{\text{down}}$ $\qquad F\sin 60^\circ = mg + T\sin 45^\circ$

$F_1 = F_1$ $\qquad F\cos 60^\circ = T\cos 45^\circ$

Divide: $$\tan 60^\circ = \frac{mg + T\sin 45^\circ}{T\cos 45^\circ}$$

$$T = \frac{mg}{\tan 60^\circ \cos 45^\circ - \sin 45^\circ} = 3790\text{ N}$$

$$F = \frac{\cos 45^\circ}{\cos 60^\circ}T = 5350\text{ N}$$

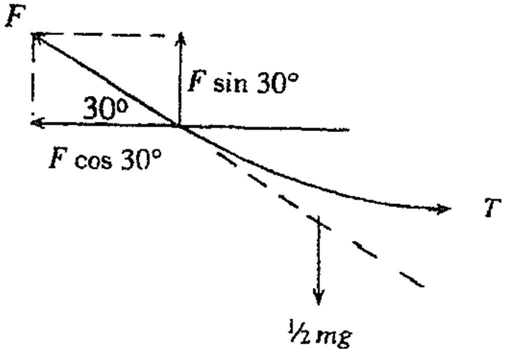


SP 5.4 Draw the force diagram for one-half of the cable.

$F_{\text{up}} = F_{\text{down}}$ $\qquad F_L = F_R$

$F\sin 30^\circ = \frac{1}{2}mg$ $\qquad 0.5F = 0.5mg$ $\qquad F = mg$

$F\cos 30^\circ = T$ $\qquad T = 0.87\,mg$



SP 5.5 Draw the force diagram for each pulley alone. Thus,

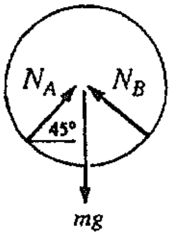
$$2T_1 = W \qquad T_1 = 0.5W$$

$$2T_2 = T_1 \qquad T_2 = 0.5T_1 = 0.25W$$

$$T_3 = T_4 = F \qquad 2T_3 = T_2 \qquad T_3 = 0.125W = T_4 = F$$

SP 5.6 By symmetry $N_A = N_B$ and $F_{\text{up}} = F_{\text{down}}$:

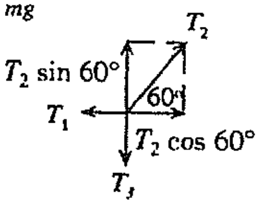
$$2N_a \sin 45° = mg \qquad N_A = N_B = \frac{1}{\sqrt{2}}mg$$



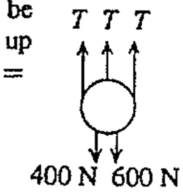
SP 5.7 $T_3 = mg \qquad T_2 \sin 60° = mg \qquad T_2 \cos 60° = T_1$

$$50T_1 = mg \tan 60° \qquad T = \frac{mg}{\sin 60°}\, T_{mg}$$

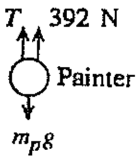
$$m = 20\,\text{kg, so } T_1 = 339\text{ N},\ T_2 = 226\text{ N},\ T_3 = 196\text{ N}$$

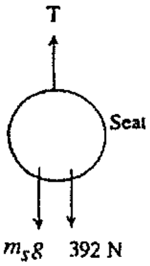


SP 5.8 Let the painter platform and lower pulley be the system. Three segments of the rope held pull up on this system, so $3T = 400\text{ N} + 600\text{ N}$, and $T = 333$ N.



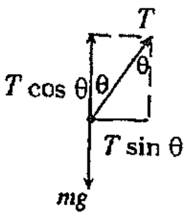
SP 5.9 The seat pushes up on the painter with force $F = 392$ N, so for the painter $T + 392 - m_p g = m_p a$. For the seat, $T - 392 - m_s g = m_s a$. Subtract these equations and solve for a: $a = 1.09\,\text{m/s}^2$, and $T = 479$ N.



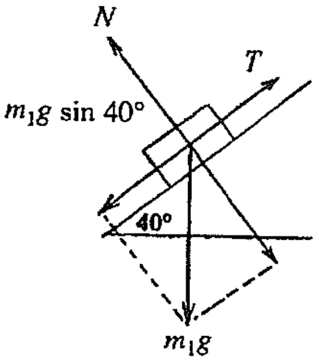


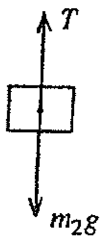
SP 5.10
Divide:

$$T \cos \theta = ma$$
$$a = g \tan \theta$$



SP 5.11 Assume m_2 accelerates down. If this is wrong, a will turn out to be negative. For m_2, $m_2 g - T = m_2 a$. For m_1, $T - mg \sin 40° = m_1 a$. Solve one equation for T and substitute in the other. Find $a = 0.86\,\text{m/s}^2$ and $T = 35.8$ N. So m_2 goes down.





SP 5.12 (a) $0.75mg - mg = ma$ so $a = -0.25g = -0.25(9.8\,\text{m/s}^2) = -2.45\,\text{m/s}^2$.
(b) $v^2 = v_0^2 + 2ay = 0 + 2(-2.45\,\text{m/s}^2)(-20\,\text{m})$, and $v = 9.9\,\text{m/s}$.
(c) Jump and $a = -g$, $v^2 = 0 + 2(-9.8\,\text{m/s}^2)(20\,\text{m})$, and $v = 19\,\text{m/s}$.

SP 5.13 (a) $v_y^2 = v_{0y}^2 - 2gh = 0$ at the highest point. $v_{0y}^2 = (v_0 \sin 60^o)^2 = 2gh$, and $h = 0.32\,m$. Find $v_0 = 2.9\,\text{m/s}$:

$$v_0 = at = \frac{F}{m}t, \qquad \text{so } F = \frac{mv_0}{t} = \left(\frac{v_0}{gt}\right)mg = 295mg\,!!!$$

(b) $a = \frac{v_0}{t} = 2.9 \times 10^3\,\text{m/s}^2 = 295g$. Amazing!

SP 5.14 Going down, $-Mg + F = -Ma$, and $F =$ lift force. Going up, $F - (M - m)g = (M - m)\,a$. Solve simultaneously for m, and find

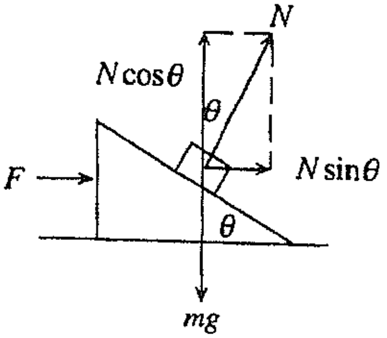
$$m = \frac{2a}{a+g}M$$

SP 5.15 $bv - mg = ma$. v increases until $bv - mg = 0$. Thus $a = 0$, and $v =$ constant when $v = mg/b$.

SP 5.16 Considering both masses as one object, $F = (m + M)a$. For the small block alone, $f_x = N \sin\,\theta = ma$ and $N \cos\,\theta = mg$.

$$\frac{N\,\sin\,\theta}{N\,\cos\,\theta} = \tan\,\theta = \frac{a}{g}, \quad a = g\,\tan\,\theta$$

$$F = (m + M)\,a = (m + M)\,g\,\tan\,\theta$$



SP 5.17 For both masses considered as the system, $T = (m + M)\,a_x$. Observe that when the base moves 1 cm to the right, mass m drops 2 cm, so for $-a_y = 2\,a_x$ (down is negative). For mass m, $T - mg = -ma_x = -2\,ma_x$. Solve for a_x:

$$a_x = \left(\frac{2m}{5m + M}\right)g = -\frac{1}{2}\,a_y$$

$$-h = \frac{1}{2}\,a_y t^2 \qquad t^2 = -\frac{2h}{a_y} = \frac{(5m + M)h}{2mg} \qquad t = \sqrt{\frac{(5m + M)h}{2mg}}$$

SP 5.18 Observe that when the wedge moves a distance x to the right, the rod moves up a distance y, when $\tan\,\theta = y/x$. Thus the vertical acceleration of the rod a_y is related to the horizontal acceleration of the wedge by $a_y = a_x \tan\,\theta$. The wedge exerts a normal force on the rod of mass m, so

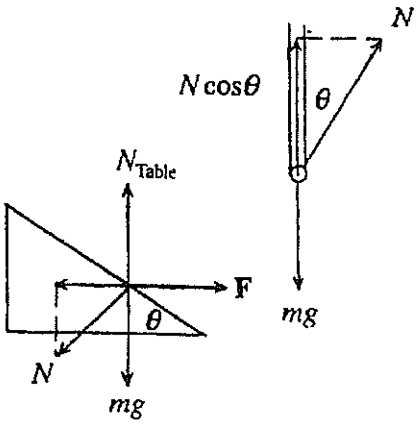
$$N\,\sin\,\theta - mg = ma_y \qquad (i)$$

The rod exerts a force N down on the wedge, so

$$F - N\,\sin\,\theta = Ma_x \qquad (ii)$$

Solve Eqs. (i) and (ii) for F, given $a_y = 2g = a_x \tan\,\theta$:

$$F = \left(3m\tan\,\theta + \frac{2M}{\tan\,\theta}\right)g = 13.3\,\text{N}$$



Chapter 6

Circular Motion

6.1 CENTRIPETAL FORCE

Whenever a moving object turns, its velocity changes direction. Since acceleration is a measure of the rate of change of velocity, an object that turns is accelerating. This kind of acceleration, called *centripetal* or *radial acceleration*, is related to v, the speed, and r, the radius of the curvature of the turn, by $a_c = v^2/r$. This relationship was obtained in Section 4.4. There we saw that there were two kinds of acceleration. Tangential acceleration (I called this "speeding up or slowing down" acceleration) measures the rate of change of speed. Radial, or centripetal, acceleration ("turning" acceleration) measures the rate of change of velocity associated with changing direction. Radial acceleration is directed perpendicular to the velocity vector and points toward the center of the arc on which the object is moving. If an object turns left, it accelerates left. If it turns right, it accelerates right.

From Newton's second law of motion we saw that in order for an object to accelerate, it must be subject to a net force. This is illustrated in Figure 6.1. The amount of force needed to cause an object with speed v to curve along an arc of radius r is thus

$$\boxed{F_c = ma_c = \frac{mv^2}{r}} \qquad (6.1)$$

Observe that F_c is not a *kind* of force. Many different kinds of forces can be used to make an object turn. For example, gravity causes the moon to curve and travel a circular path around the earth. The tension in a rope causes a tether ball to travel in a circle. The normal force exerted by a banked curve on a highway causes a car to travel a circular path. The friction force between a car's tire and the roadway causes the car to turn. When you draw a force diagram, *do not draw in a force labeled F_c*. Centripetal force is a way of using a force, not a kind of force.

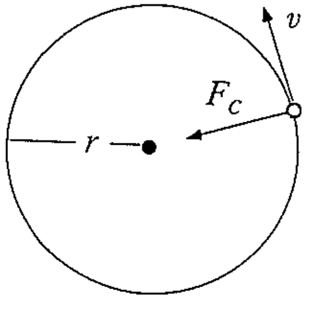


Figure 6.1

An object does not have to travel in a complete circle to experience centripetal acceleration. However, frequently objects do travel around and around, as is the case with a spinning wheel or compact disk. Suppose that an object makes f rev/s. The

frequency of revolution is f. One revolution per second is called **1 hertz** (a dumb way of labeling something, but we're stuck with it). Later we will extend this idea of frequency to anything that varies periodically, whether or not it moves in a circle. Thus the electricity in your house varies at 60 times per second, or 60 Hz. The AM radio station in my town broadcasts radio waves at a frequency of 1400 kHz. (The announcer always says, "KRPL at 1400 on your AM dial." He means 1,400,000 variations per second in the radio wave electric field.)

In one revolution an object travels a distance $2\pi r$. If it makes f rev/s, the distance traveled in 1 s is $2\pi r f$. The distance traveled per second is the speed v, so

$$v = 2\pi r f \tag{6.2}$$

One revolution is 2 radians (rad). The number of radians swept out per second is called the **angular frequency** or **angular velocity**, in **radians per second** (both terms are used). This symbol ω that looks like a small w is a lowercase Greek omega. It is measured in radians per second. Thus

$$\boxed{\omega = 2\pi f} \tag{6.3}$$

and

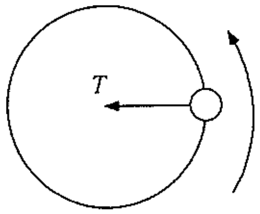
$$\boxed{v = r\omega} \tag{6.4}$$

In terms of ω the centripetal force can be written

$$\boxed{F_c = \tfrac{mv^2}{r} = mr\omega^2} \tag{6.5}$$

When dealing with objects rotating at constant frequency, it is easiest to use Eq. 6.5. When an object is simply turning, such as a jet plane pulling out of a dive, use Eq. 6.1.

Problem 6.1 A ball of mass 0.15 kg slides with negligible friction on a horizontal plane. The ball is attached to a pivot by means of a string 0.60 m long. The ball moves around a circle at 10 rev/s. What is the tension in the string?



Solution The ball stays in the horizontal plane, so the upward normal force exerted by the table balances the downward force of gravity (the weight of the ball). In the horizontal plane only one force acts on the ball, the tension of the string. Thus a net force is acting on the ball, and it is not in equilibrium. It is accelerating with acceleration a_c. The force required to cause this acceleration is

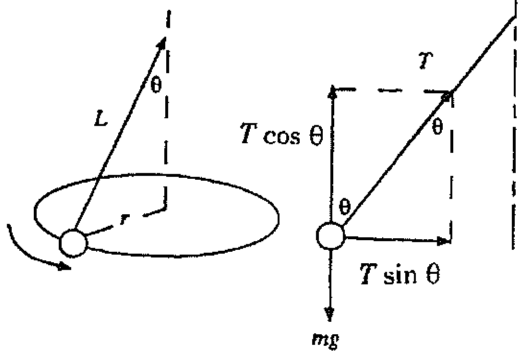
$$F_c = T = mr\omega^2 = (0.15\,\text{kg})(0.60\,\text{m})\left(\tfrac{2\pi\times 0.1}{\text{s}}\right)^2 = 14\,\text{N}$$

Note that the units for ω and f are s^{-1}. Radians and revolutions are not "units" as such.

Problem 6.2 In a popular carnival ride a person sits in a chair attached by means of a cable to a tall central post. The pole is spun, causing the rider to travel in a horizontal circle, with the cable making an angle θ with the vertical pole. A contraption like this is called a *conical pendulum*. Suppose the rider and chair have mass of 150 kg. If the cable length is 8 m, at what frequency should the chair rotate if the cable is to make an angle of 60° with vertical? What is the tension in the cable?

Solution Always begin by drawing the force diagram, as shown here. The rider is not moving up or down, so the vertical forces are in balance.

$$F_{\text{up}} = F_{\text{down}} \qquad T\cos\theta = mg$$



The component of the cable tension T directed horizontally, that is, toward the center of rotation, is a net force causing a centripetal acceleration toward the center. The magnitude of this required net force is given by Eq. 6.5: $T\sin\theta = mr\omega^2$, where $r = L\sin\theta$. Thus $T\sin\theta = mL\sin\theta\,\omega^2$, $T = mL\omega^2$, and

$$T = \frac{mg}{\cos\theta} = \frac{(150\text{ kg})(9.8\text{ m/s}^2)}{\cos 60°} = 2950\text{ N}$$

$$\omega = 2\pi f = \sqrt{\frac{T}{mL}}$$

so

$$f = \frac{1}{2\pi}\left[\frac{2940N}{(150\text{ kg})(8\text{ m})}\right]^{1/2} = 0.25\text{ Hz}$$

Problem 6.3 On a level roadway the coefficient of friction between the tires of a car and the asphalt is 0.80. What is the maximum speed at which a car can round a turn of radius 25 m if the car is not to slip?

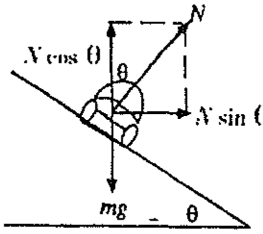
Solution The force of friction provides the needed turning force F_c. Thus

$$F_f = \mu N = \mu mg = \frac{mv^2}{r}$$

$$v^2 = \mu rg = (0.08)(25\text{ m})(9.8\text{ m/s}^2) \qquad v = 14\text{ m/s} = 31\text{ mi/h}$$

Problem 6.4 A car traveling on a freeway goes around a curve of radius r at speed v. The roadway is banked to provide the necessary inward centripetal force in order for the car to stay in its lane. At what angle should the roadway be banked if the car is not to utilize friction to make the turn?

Solution Draw the force diagram. The car is not accelerating in the vertical direction, so $F_{\mathrm{up}} = F_{\mathrm{down}}$, and $N \cos \theta = mg$. The horizontal component of the normal force N provides the needed centripetal force:



$$N \sin \theta = \frac{mv^2}{r}$$

Divide: $$\frac{N \sin \theta}{N \cos \theta} = \frac{mv^2}{mgr} \qquad \tan \theta = \frac{v^2}{rg}$$

Caution: You might be tempted to resolve the weight and the normal force into components parallel and perpendicular to the road surface. You might then imagine that the components perpendicular to the surface are in balance. This is wrong because the road surface is not truly directed perpendicular to the plane of the paper. Our drawing is somewhat misleading in this respect. The normal force is actually larger than the car's weight since it must support the weight and also provide an inward force to make the car turn.

Problem 6.5 In the "Human Fly" carnival ride a bunch of people stand with their backs to the wall of a cylindrical room. Once everyone is in place, the room begins to spin. The inward normal force exerted by the wall on the back of each person provides the needed centripetal force to ensure that each person travels in a circle of diameter equal to the diameter of the room. Once the room is spinning rapidly, the floor drops out from beneath the people. Friction between the wall and each person's back "glues" each one to the wall, although with some effort they can squirm and move about (like human flies on a wall). Personally, this is not my cup of tea. I get motion sickness, but kids love it. What minimum coefficient of friction is needed if the people are not to slip downward, assuming the room diameter is 4.0 m and the room spins at 18 rev/min?

Solution For no slipping down, $F_f = mg$ and $N = mr\omega^2$.

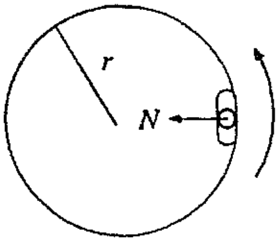
$$\omega = 2\pi f \qquad F_f = \mu N$$

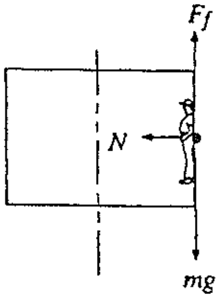
Thus $\mu m r\omega^2 = mg$:

$$\mu = \frac{g}{r\omega^2} = \frac{g}{r(2\pi f)^2}$$

$$= \frac{9.8\ \mathrm{m/s}^2}{(4\ \mathrm{m})(2\pi)^2[(18/60)\mathrm{s}^{-1}]^2}$$

$$= 0.69$$





Problem 6.6 An F14 jet fighter traveling 260 m/s (about 580 mi/h) pulls out of a vertical dive by turning upward along a circular arc of radius 2.4 km. What acceleration does the

pilot experience? Express the result as a multiple of g. If the pilot's weight is $560N$, what force does the seat exert on him? (This is his "apparent weight.") Even with a pressurized suit, the maximum acceleration a person can experience without suffering brain hemorrhaging resulting in a blackout is about $11g$. Thus a pilot must avoid turning too sharply so that he or she won't black out and the wings won't break off the airplane.

Solution $$a_c = \frac{v^2}{r} = \frac{(260\,\mathrm{m/s})^2}{2400\,\mathrm{m}} = 28.2\,\mathrm{m/s^2} = \frac{28.2}{9.8}g = 2.87g$$

Here g is the acceleration due to gravity. At the lowest point in the dive, gravity acts downward on the pilot with force W, and the seat pushes up with a normal force N, so

$$N - W = ma_c = \frac{mv^2}{r} = \frac{mg}{g}\frac{v^2}{r} = W\frac{a}{g}$$

so $$N = W + W(\tfrac{a}{g}) = W(1+\tfrac{a}{g}) = 560(1+2.87)N$$

$$= 3.87W = 2090N$$

Problem 6.7 A woman stands a distance of 2.40 m from the axis of a rotating merry-go-round platform. The coefficient of friction between her shoes and the platform surface is 0.60. What is the maximum number of revolutions per minute the merry-go-round can make if she is not to start slipping outward?

Solution Friction provides the needed inward centripetal force, so $F_c = mr\omega^2 = F_f$. $F_f = \mu mg$, so

$$\omega^2 = (2\pi f)^2 = \frac{\mu mg}{mr} \qquad f = \frac{1}{2\pi}\sqrt{\frac{(0.60)(9.8\,\mathrm{m/s^2})}{2.40\,\mathrm{m}}}$$

$$f = 0.25\,\mathrm{s^{-1}} = (60)(0.25)\,\mathrm{rev/min} = 15\,\mathrm{rev/min}$$

Problem 6.8 Suppose you are driving at speed v_0 and find yourself heading straight for a brick wall that intersects the line of your path at 90°. Assuming that the coefficients of friction for stopping and for turning are the same, are your chances of avoiding a crash better if you continue straight ahead while braking or if you simply turn along a circular path at a constant speed?

Solution The braking force is $F_f = \mu mg$, so the braking acceleration is $F_f/m = -\mu g$. From Eq. 3.9, $v^2 = v_0^2 + 2\,ax = 0$ when stopped. Thus the stopping distance is

$$x = -\frac{v_0^2}{2a} = \frac{v_0^2}{2\mu g}$$

For turning, friction provides the centripetal force, so

$$\frac{mv_0^2}{r} = \mu mg \qquad r = \frac{v_0^2}{\mu g}$$

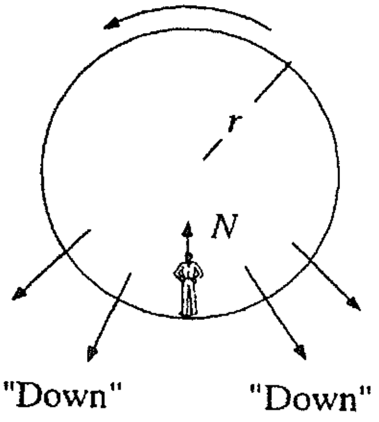
Thus $x < r$, and it is better to brake than to turn.

Problem 6.9 The first space habitats built by humans will most likely be cylindrical in shape. It is envisioned that such a space habitat will be rotated about the axis of the cylinder in order to simulate the effect of gravity here on earth. Thus an inhabitant will feel the floor pushing on his or her feet with a force of $N = mr\omega^2$ in order to cause him or her to move along a circular path as the space cylinder rotates. He or she will then experience a sort of "artificial gravity" pulling downward to counter the upward force of the floor. (Downward will be radially out.) Preliminary NASA designs have been developed for a cylinder about 6.4 km in diameter and 32 km in length. Later much larger structures could be built. At what rate would such a structure have to rotate in order to simulate the same acceleration due to gravity found here on earth?

Solution We require $mg = mr\omega^2$ so

$$\omega = 2\pi f = \sqrt{\frac{g}{r}}$$

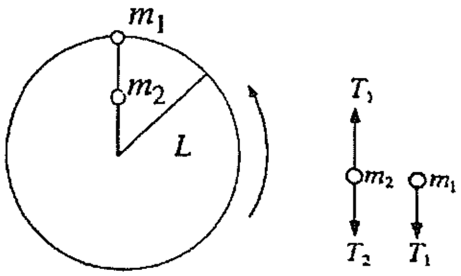
$$f = \frac{1}{2\pi}\sqrt{\frac{9.8\,\text{m/s}^2}{6400\,\text{m}}} = 0.006\ \text{s}^{-1} = 0.37\ \text{rev/min}$$



Note that your apparent weight $mr\omega^2$ decreases as r becomes smaller, that is, as you approach the axis of the cylinder. This could have important applications. For example, it is very difficult to hospitalize severely burned patients since they must lie on open wounds. Near the center of the space habitat, a person would feel"weightless" and could simply float above a bed with very little support.

Problem 6.10 The inward centripetal force required to cause any object to move in a circle can be very large when rigid objects are rotated at high frequency, as is the case in most machines. Spinning gears and wheels can be subject to huge forces that can cause them to fracture and cause serious damage. My father, a machinist, lost an eye when a grinding wheel he was using fractured and sent fragments in all directions. Some of the pieces struck his face. Experimental cars have been designed that are propelled by the energy stored in spinning flywheels (as an alternative to using internal combustion engines), but a limiting factor in the use of such machines is their ability not to fracture when rotated at high speed.

To gain an understanding of the forces involved when objects rotate, consider the following simple model. A very light rod of length L is rotated in a horizontal plane with one end fixed. At one end is attached a mass m_1, and at the center of the rod is attached a mass m_2. Determine the tension T_1 in the portion of the rod between $m_1$and m_2 and the tension in the rod between m_2 and the axis of rotation.



Solution The forces acting on m_1 and m_2 are drawn here. Note that the tension T_1 in the outer portion of the rod pulls *inward* on m_1 and *outward* on m_2. There must be a net inward centripetal force on each mass, so $T_2 > T_1$. Applying $F_c = ma_c$ to each mass, $T_1 = m_1r_1(2\pi f)^2$. $T_2 - T_1 = m_2r_2 (2\pi f)^2$, where $r_1 = L$, and $r_2 = 1/2\ L$. Thus $T_1 = m_1L\,(2\pi f)^2$ and

$$T_2 - mL\,(2\pi f)^2 = m_2\tfrac{L}{2}(2\pi f)^2$$

If $m_1 = m_2$, we see that $T_1 = 2T_2$. The tension is greater farther out, and it increases as the square of the frequency of rotation. Always wear eye protection when working around equipment or machinery with rotating parts.

6.2 SUMMARY OF KEY EQUATIONS

$$\omega = 2\pi f \qquad s = r\theta \qquad a_c = \tfrac{v^2}{r} = r\omega^2 \qquad F_c = m\tfrac{v^2}{r} = mr\omega^2$$

Supplementary Problems

SP 6.1 The silt particles in lake water gradually settle to the bottom of the lake due to the force of gravity on them. The rate of this sedimentation depends on the size and shape of the particles and on the strength of the gravity force acting on them. This process can be very slow. A centrifuge is a laboratory device widely used in biological science to isolate macromolecules like nucleic acids (for example, DNA) or proteins by sedimentation. The molecules are in liquid in a test tube that is placed in an ultracentrifuge. This apparatus rotates the test tube at high frequency, and the inward centripetal force exerted by the bottom of the test tube, $mr\,(2\pi f)^2$, produces an artificial gravity force mg' where $g' = r\,(2\pi f)^2$. By rotating the sample at very high frequency the "effective g," g', can be made very large, and the resulting sedimentation time can be made short enough to separate out various macromolecules in reasonable times (instead of the thousands of years that would be required using normal gravity). What is the effective g' obtained when a sample is rotated at 70,000 rev/min at a distance of 5 cm from the axis?

SP 6.2 In an enduro motorcycle race a rider goes over the top of a small hill that is approximately spherical with a radius of curvature of 12 m. What is her maximum speed if she is not to become airborne?

SP 6.3 We saw in Problem 6.8 that when approaching a wall head on in your car at speed v_0, it is better to brake than to turn in order to avoid colliding with the wall. Suppose, however, that your line of motion intersects the plane of the wall at an angle θ. For what value of θ are your chances of barely avoiding a collision equal whether you brake or turn?

SP 6.4 Suppose the earth is a sphere of radius 6370 km. If a person stood on a scale at the north pole and observed the scale reading (her weight) to be mg, what would the scale read if she stood on it at a point on the equator?

SP 6.5 A small block of mass m slides with negligible friction in a horizontal circle on the inside of a conical surface. The axis of the cone is vertical, and the half angle of the cone is 60°. The block rotates at 1.20 rev/s. How high above the apex of the cone does the block slide?

SP 6.6 In a ball mill used to polish stones, a cylindrical can of radius r is oriented with its axis horizontal and rotated at frequency f. The stones to be polished are immersed in a slurry containing grit and placed in the can. The rate of rotation of the can is chosen so that the stones fall away from the wall at an optimum position (determined experimentally). This same principle is used in a clothes dryer that tumbles wet clothes in order to dry them. Suppose it is desired to design a ball mill in which a rock will fall away from the wall at a given angle θ, where θ is the angle between vertical and the radius line from the axis of the cylinder to the rock. At what frequency should the can be rotated to achieve this result?

Solutions to Supplementary Problems

SP 6.1 $g' = a_c = r\,(2\pi f)^2 = (0.05\text{ m})(2\pi)^2\left(\frac{70{,}000}{60\text{ s}}\right)^2$

$$= 2.69\ \times 10^{6}\text{ m/s}^2 = \frac{2.69 \times 10^6}{9.8} g = 2.74 \times 10^{5} g$$

SP 6.2 At the maximum allowable speed, the ground is not pushing up at all on the motorcycle, so the gravity force mg provides the needed centripetal force to keep the motorcycle moving along the circular arc of the hill's surface. Thus,

$$mg = m\frac{v^2}{r} \qquad \text{so } v^2 = rg = (12\text{ m})(9.8\text{ m/s}^2), \qquad v = 10.8\text{ m/s} \qquad \text{(about 24 mi/h)}$$

SP 6.3 The two possible paths, braking or turning, are shown here. The braking distance is s, given by

$$v^2 = v_0^2 - 2\,as = 0, \qquad s = \frac{v_0^2}{2a}$$

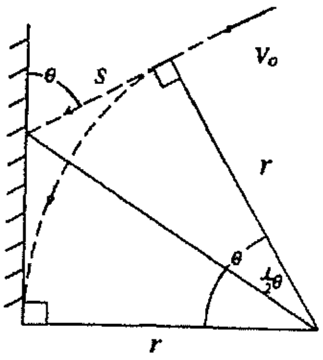
$$a = \frac{F_f}{m} = \frac{\mu mg}{m} = \mu g, \qquad \text{so } s = \frac{v_0^2}{2\mu g}$$

For turning,

$$\frac{mv_0^2}{r} = F_f = \mu mg, \qquad \text{so } r = \frac{v_0^2}{\mu g}$$

From the drawing I see that $s/r = \tan\theta/2$, so

$$\frac{v_0^2/2\mu g}{v_0^2/\mu g} = \frac{1}{2} = \tan\frac{\theta}{2} \qquad \frac{\theta}{2} = 26.6^\circ \qquad \theta = 53.1^\circ$$



SP 6.4 The gravity force mg acts on the person in a direction toward the center of the earth, and the normal force N exerted by the scale acts radially outward. The net force inward must have a magnitude of $mr\omega^2$. Thus

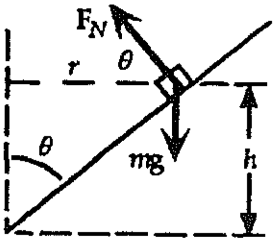
$$mr\omega^2 = mg - N \qquad N = mg - mr\omega^2 = mg - mr(2\pi f)^{\,2} = mg - mr\,\frac{2\pi}{T}^2$$

where $T = \frac{1}{f} = 24\,\text{h} = mg - mg\frac{r}{g}\left(\frac{2\pi}{T}\right)^2 = mg\left[1 - \frac{r}{g}\left(\frac{2\pi}{T}\right)^2\right] = mg\left[1 - \frac{6.37 \times 10^6}{9.8}\left(\frac{2\pi}{(24)(3600)}\right)^2\right]$

$$N = 0.997mg$$

Thus a person's apparent weight at the equator would be slightly less than his or her true weight.

SP 6.5 The inward component of the normal force exerted by the conical surface provides the needed centripetal force, and the vertical component of the normal force balances the downward force of gravity.



$$N \sin\theta = mg \qquad N\cos\theta = mr(2\pi f)^2$$

Divide:

$$\frac{\sin\theta}{\cos\theta} = \frac{mg}{mr(2\pi f)^2}$$

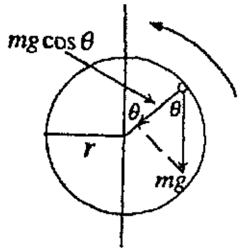
$$r = \frac{g}{(\tan\theta)(2\pi f)^2} \qquad h = \frac{r}{\tan\theta} = \frac{g}{(2\pi f)^2}\frac{1}{\tan^2\theta}$$

$$g = 9.8\,\text{m/s}^2 \qquad f = 1.20\,\text{s}^{-1} \qquad \theta = 60°$$

$$h = 0.057\,\text{m}$$

SP 6.6 At the moment the rock falls away from the wall, the inward normal force exerted on the rock by the wall drops to zero. Thus at this instant the inward centripetal force acting on the rock is just the inward component of the gravity force $mg\sin\theta$. Thus,

$$mr(2\pi f)^2 = mg\sin\theta, \quad \text{so } f = \frac{1}{2\pi}\sqrt{\frac{g\sin\theta}{r}}$$



Chapter 7

Work and Energy

We have seen how to use displacement, velocity, acceleration, and force to describe the behavior of some simple mechanical systems. With this background we now develop an alternate approach based on Newton's laws that is simpler and very widely applicable. The concepts of work and energy will enable us to extend our previous analysis to complicated systems like the human body, the ecosystem in which we live, and chemical and nuclear reactions.

7.1 WORK

You probably have a pretty good idea of what is meant by the term work. When you push on a heavy packing crate and slide it across the floor, you do work on it. You get tired because you have used energy to do the work. Suppose a constant force F_x acts in the x direction and causes an object to move a distance of Δx. We define the work done by the force as

$$\boxed{W = F_x\,\Delta x} \tag{7.1}$$

Since work is the product of force and distance, it is measured in units of newton-meters. Work is such an important concept that its unit is given a special name, the **joule.** 1 newton-meter (N · m) = 1 joule (J). There are two important points worth recognizing here. First, in order for a force to do work on an object, the object must move. If you stand holding motionless a heavy concrete block in your hands, you may get tired, but you are not doing work on the block. Second, only forces directed along the line of motion of an object do work on the object. When you push a crate across the floor, the force of gravity does no work on the crate because the gravity force is directed downward, perpendicular to the direction of motion of the crate.

Consider a constant force F that acts on an object that is moved. If the force makes an angle θ with the direction of motion (which I take to be the x axis), the work done by F is

$$\boxed{W = F_x\,\Delta x = F\cos\,\theta\,\Delta x} \tag{7.2}$$

Observe that when $90^o < \theta < 180^o$, $\cos\,\theta\, < 0$ and the work done by F is negative. What this means is that instead of the force doing work on the moving object, the object is doing work on whatever generates the force. Some books use this idea of negative work, but it is not necessary to do so, and I find the idea needlessly confusing. It is sort of like saying that when you draw money out of the bank, you make a negative deposit. I like to keep things simple.

Problem 7.1 A logger drags a heavy log across level ground by attaching a cable from the log to a bulldozer. The cable is inclined upward from horizontal at an angle of 20°. The cable exerts a constant force of $2000 N$ while pulling the log 16 m. How much work is done in dragging the log?

Solution $W = F \cos\theta\, \Delta x = (2000\, N)(\cos 20°)(16\,\text{m}) = 3.0 \times 10^4\ \text{J}$

If the displacement Δx is written as a vector, we can write Eq. 7.2 in the following useful form:

$$\boxed{W = F \cos\theta\, \Delta x = \mathbf{F} \cdot \Delta \mathbf{x}} \qquad (7.3)$$

This is an important relation. **MEMORIZE IT.**

Frequently the forces we encounter are not constant. They change in strength and direction as the object on which they act moves. For example, suppose a force $F(x)$ that is a function of position acts on an object that moves from position x_1 to position x_2. We can imagine this displacement to consist of many small steps Δx_i. The total work done is the sum of the work done to make each little step.

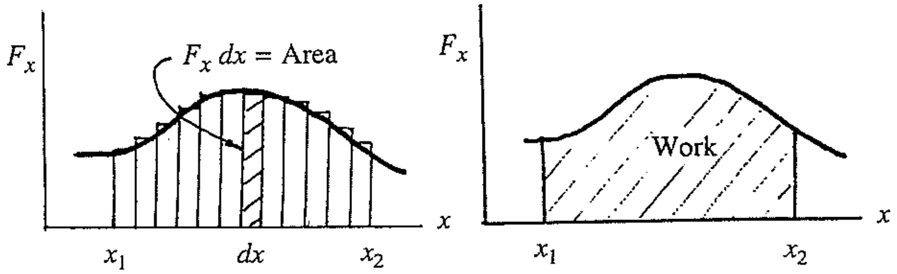
$$W = \sum F(x)\ \Delta x_i$$

In the limit that $\Delta x \to 0$, this may be written as an integral.

$$\boxed{W = \int_{x_1}^{x_2} F(x)\ dx} \qquad (7.4)$$

Equation 7.4 has a simple (and useful) interpretation in terms of a graph of force versus displacement, as illustrated in Figure 7.1. The work done in each little step dx is just $F_x\ dx$. But $F_x\ dx$ is the area of the small shaded rectangle in the drawing. The total work done in going from x_1 to x_2 is thus seen to be equal to the area under the force versus displacement curve.

Figure 7.1



An important example of a nonconstant force is the force exerted by a spring or a rubber band. Suppose a mass is attached to one end of a spring and placed on a frictionless horizontal surface. The other end of the spring is attached to a fixed point. Take the position of the mass when the spring is unstretched to be $x = 0$. If the mass is then displaced an amount x from its equilibrium position, the spring exerts a force F on it, where

$$\boxed{F = -kx} \tag{7.5}$$

This kind of force is called a **Hooke's law force,** and k is the **spring constant.** The minus sign indicates that this spring force is a **restoring force**. That means that the spring force always tries to make the attached mass move back toward its equilibrium position. When x is positive, F is negative. When x is negative, F is positive. This is illustrated in Figure 7.2.

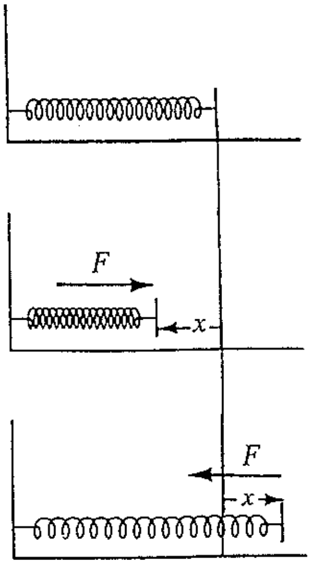


Figure 7.2

This Hooke's law force is a big deal because it is a first approximation to the forces we encounter in many situations, for example, the forces on a girder in a bridge or between ions in a crystal. We will study it in greater detail later.

If you want to stretch the spring from its equilibrium position (at $x = 0$) to a point where it is stretched an amount x, you must exert a force opposite to the force exerted by the spring. Of course, to actually get the mass to move from rest, you would have to exert a force a teensy bit bigger than the spring force, but if you don't mind taking forever to move the mass, a force equal in strength to the spring force will suffice. Hence the work done in stretching a spring a distance x is

$$W = \int_0^x F\ dx = \int_0^x kx\ dx$$

$$\boxed{W = \frac{1}{2}\, kx^2} \tag{7.6}$$

Problem 7.2 A force of 120 N will stretch a spring 2 cm. What is the spring constant of the spring? If the spring were cut in half, what would then be the spring constant?

Solution $F = -kx$. k is always positive since F and x always have opposite signs. Thus the magnitude of k is

$$k = \frac{120\,\text{N}}{0.02\,\text{m}} = 6000\,\text{N/m}$$

If the entire spring stretched by 2 cm, half of its length stretched by 1 cm, whereas the force acting on it is still 120 N, so the spring constant for this smaller spring (half the original one) is 12,000 N · m. Thus the small spring is stiffer than the longer one. You can test this for yourself by joining rubber bands together. The longer the chain of rubber bands, the "softer" the spring they make. Note that the tension in a spring, like the tension in a light string, is the same everywhere.

Problem 7.3 How much work must be done to stretch a spring by 2 cm if the spring constant is 640 N · m?

Solution $$W = \frac{1}{2}\,kx^2 = \frac{1}{2}\,(640\,\mathrm{N}\cdot\mathrm{m})(0.02\,\mathrm{m})^2 = 0.13\,\mathrm{J}$$

7.2 KINETIC ENERGY

Suppose that a single constant force F acts on a particle in its direction of motion and causes it to accelerate, increasing the speed from an initial value v_0 up to a final value v. Recall that for an object with constant acceleration, $v^2 = v_0^2 + 2\,ax$. Substitute $a = F/m$. Then

$$v^2 = v_0^2 + \frac{2\,Fx}{m}$$

The work done by the force is $W = Fx$, so

$$\boxed{W = \frac{1}{2}\,mv^2 - \frac{1}{2}mv_0^2} \tag{7.7}$$

This is a very important equation. **MEMORIZE IT.**

We define the **kinetic energy** of an object of mass m with speed v as

$$\boxed{KE = \frac{1}{2}\,mv^2} \tag{7.8}$$

Energy, like work, is measured in joules: Equation 7.7 is the **work-energy theorem**. It says that **the work done on a particle is equal to the increase in the kinetic energy of the particle.** The work-energy theorem is valid even if the force is varying. Thus

$$W = \int_{x_0}^{x} F_x\; dx = \int_{x_0}^{x} ma\; dx = \int_{x_0}^{x} m\frac{dv}{dt}\; dx = \int_{v_0}^{v} m\frac{dx}{dt}\; dv = \int_{v_0}^{v} mv\; dv$$
$$= \frac{1}{2}\,mv^2 - \frac{1}{2}\,mv_0^2$$

In the discussion above I imagined an external force acting on a particle to speed it up and increase its kinetic energy. I could equally well consider a case in which a particle experiences a retarding force that slows it down. In this case the kinetic energy would decrease, and the particle would do work on an external entity. Using the analysis above, we could deduce that the loss in kinetic energy would be equal to the work done by the particle. This is what happens in your car engine when a rapidly moving gas molecule strikes the piston and pushes on it. The gas molecule slows down and in so doing, does work on the piston, and this work is then transferred into making the car move forward. This observation leads us to a pretty good working definition of the meaning of *energy.*

DEFINITION: The energy of a system is a measure of its ability to do work.

Energy can also be used to change the state of matter, for example, to melt a solid into a liquid. Energy takes many forms. Kinetic energy ("motion energy") is just one form of energy. Other forms of energy include light, thermal energy, chemical energy, mass energy (sometimes called "nuclear energy"), electric energy, magnetic energy, gravitational energy, and sound. These are only rough descriptors, and some of the categories overlap. Some types of energy are grouped under the broad title "potential energy," a term used to describe energy when it is more or less "stored" for future use. Thermal energy includes the kinetic energy of moving atoms as well as the stored potential energy associated with stretched electric bonds between atoms.

Problem 7.4 Calculate the kinetic energy of each of the following:

(a) The earth orbiting the sun	$m = 5.98 \text{ x } 10^{24}$ kg	$v = 2.98 \times 10^4$ m/s
(b) Car driving 60 mi/h	$m = 1500$ kg	$v = 27$ m/s
(c) World-class sprinter	$m = 80$ kg	$v = 10$ m/s
(d) Rifle bullet	$m = 0.01$ kg	$v = 1000$ m/s
(e) Nitrogen molecule in air	$m = 4.6 \text{ x } 10^{-26}$ kg	$v = 500$ m/s

Solution Using $KE = 1/2\, mv^2$ yields the following interesting results:

(a) 2.66×10^{33} J (b) 5.47×10^5 J (c) 4×10^3 J (d) 5×10^3 J (e) 5.8×10^{-21} J

7.3 POWER

DEFINITION: Power is the rate of doing work or of transferring energy.

If work dW is done in time dt, the instantaneous power used is

$$\boxed{P = \frac{dw}{dt}} \tag{7.9}$$

If a hot object radiates away energy dE in time dt, the power it radiates is

$$\boxed{P = \frac{dE}{dt}} \tag{7.10}$$

Power is measured in units of joules per second (J/s). The concept of power is so important that the unit of power is given its own name, the watt. 1 W = 1 J/s. Other commonly encountered power units are the microwatt ($1\ \mu W = 10^{-6}$ W), the milliwatt ($1\ mW = 10^{-3}$ W), the kilowatt ($1\ kW = 10^3$ W), and the megawatt ($1\ MW = 10^6$ W). The British system of units uses the horsepower unit: 1 hp = 746 W.

If the power is constant, we may write $W = Pt$ or $E = Pt$. When utility companies sell electric energy, they measure the energy sold in a unit called the kWh. They do this because a joule is a very small and inconvenient unit for their purposes.

$$1\ \text{kWh} = (1000\ \text{J/s})(3600\ \text{s}) = 3.6 \times 10^6\ \text{J}$$

Note that a watt is a unit of power that does not depend on the kind of work being done or on the kind of energy transferred. You may associate the term "watt" with electricity (as in a 100-W light bulb), but this unit applies to all kinds of work and energy. Historically people were slightly confused and used different units for different kinds of energy. For example, in chemistry and nutrition we encounter energy measured in calories or kilocalories (1 cal = 4.186 J). Architects use British thermal units (Btu's) per hour to characterize building heating systems (1 Btu = 1054 J). Electronic engineers measure energy levels in a crystal using the electronvolt ($1\ \text{eV} = 1.6 \times 10^{-19}$ J).

If force $\mathbf{F}$ causes a particle to undergo a displacement $\mathbf{ds}$, the work done is $dW = \mathbf{F} \cdot \mathbf{ds}$. Since $\mathbf{ds}/dt = \mathbf{v}$, the power provided by the force is

$$P = \frac{dW}{dt} = \frac{\mathbf{F} \cdot \mathbf{ds}}{dt} \qquad \text{or} \qquad \boxed{P = \mathbf{F} \cdot \mathbf{v}} \tag{7.11}$$

Problem 7.5 Consider a car traveling at a steady speed of 60 km/h (16.7 m/s). It encounters a frictional force (rolling and air drag) of 520 N. At what power level does the engine deliver energy to the wheels?

Solution

$$P = Fv = (520\ \text{N})(16.7\ \text{m/s}) = 8.68\ \text{kW}$$

$$= (8.68\ \text{kW})\left(\frac{1}{0.746}\,\frac{\text{hp}}{\text{kW}}\right) = 11.6\ \text{hp}$$

Note: It is not uncommon to see car ads that tout an engine rated at something like 150 hp. However, only about 20 to 25 percent of this power output is delivered to the wheels. The rest is lost as heat. Further, this high-power output is obtained only when the engine is running at full throttle. A usable power level of 10 to 15 hp is about what is required to propel a car traveling a level road at moderate speed.

Problem 7.6 Combustion of 1 gal of gasoline yields 1.3×10^8 J. Consider a car that can travel 28 mi/gal at a speed of 90 km/h. Of the energy obtained from burning the gasoline, 25 percent goes into driving the car, with the rest dissipated as heat. What is the average frictional force acting on the car?

Solution The time for the car to travel 28 mi is

$$t = \frac{(28\,\text{mi})(1.6\,\text{km/mi})}{90\,\text{km/h}} = 0.5\,\text{h}$$

$$= 1800\,\text{s}$$

The gasoline energy used in this time is that from 1 gal, 1.3×10^8 J. Of this energy, 25 percent powers the wheels, so

$$P = (0.25)\frac{(1.3\times10^8\,\text{J})}{1800\,\text{s}} = 18\,\text{kW}$$

$$v = 90\,\text{km/h} = 90\,\frac{10^3\,\text{m}}{3600\,\text{s}} = 25\,\text{m/s} \qquad F = \frac{P}{v} = \frac{18{,}000\,\text{W}}{25\,\text{m/s}} = 720\,\text{N}$$

Problem 7.7 The first human-powered airplane to cross the English Channel was the Gossamer Albatross. A person turned the propeller by means of a bicycle pedaling mechanism. In order to keep the plane flying, he had to deliver 0.3 hp to the drive mechanism throughout the duration of the 2 h, 49 min flight. Human muscles have an efficiency of about 20 percent; that is, 20 percent of the energy released in your body goes into doing mechanical work. (a) What total energy did the pilot use during this flight? (b) A Big Mac hamburger provides about 500 kcal. How many such hamburgers would the pilot have to eat to obtain enough energy for the flight across the channel?

Solution (a) $t = 2\text{ h } 49\text{ min} = (2)(3600\text{ s}) + (49)(60\,\text{s}) = 10{,}140\text{ s}$

$$W = Pt = (0.3\text{ hp})(746\ W/\text{hp})(10{,}140\,\text{s}) = 2.27 \times 10^6\text{ J}$$

If the pilot used energy E, then 20 percent of this energy went into the work of pedaling. Thus

$$0.2E = 2.27 \times 10^6\text{ J} \qquad \text{and} \qquad E = 1.13 \times 10^7\text{ J}$$

(b) 1 kcal = 4.2 kJ so the number of burgers required is

$$n = \frac{1.13\times10^7\text{ J}}{(5\times10^5\text{ cal})(4.2\text{ J/cal})} = 5.4$$

Problem 7.8 A girl riding her bicycle on a level road is traveling at 20 km/h while sitting upright. She finds that when she leans low over the handlebars, while still doing work at the same rate, that her speed increases to 22 km/h. By what factor did she reduce the drag force acting on her by leaning low over the handlebars?

Solution She maintained power constantly, so $P = F_1 v_1 = F_2 v_2$. Thus

$$F_2 = \frac{v_1}{v_2} \qquad F_1 = \frac{20}{21} \qquad F_1 = \frac{20}{22} F_1 \qquad F_2 = 0.91 F_1$$

Problem 7.9 In terms of energy consumption, walking and bicycling are much more efficient means of transportation than traveling by car or airplane. A hiker can cover about 30 mi with an additional food intake of 2000 kcal (above what is needed for sitting in one place). A bicyclist can travel about 100 mi with this same amount of supplementary food intake. One gallon of gasoline releases about 1.3×10^8 J of energy. Calculate about how many "miles per gallon" (mi/gal) a hiker and a bicyclist would get if they used gasoline as a source of energy. An average car gets about 15 to 20 mi/gal.

Solution 2000 kcal is equivalent to G gal of gas, where

$$G = \frac{(2000\ \text{cal})(4.2\ \text{J/cal})}{1.3 \times 10^8\ \text{J}} = 0.06\ \text{gal}$$

For a hiker,

$$\text{mi/gal} = \frac{30\ \text{mi}}{0.06\ \text{gal}} = 470\ \text{mi/gal}$$

For a bicyclist,

$$\text{mi/gal} = \frac{100\ \text{mi}}{0.06\ \text{gal}} = 1560\ \text{mi/gal}$$

Problem 7.10 A car of mass 1600 kg with its gears in neutral is observed to reach a constant terminal speed of 110 km/h after coasting a long way down a 10 percent grade. (This is a slope that drops 1 m for every 10 m traveled along the roadway.) The car experiences a frictional drag force bv^2. What power would be required to drive this car at 90 km/h on a level highway?

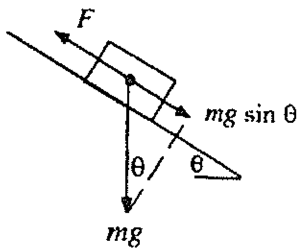
Solution When coasting at constant speed downhill, the downhill component of the gravity force $mg \sin\theta$ just balances the friction force F.

$$v = 110\ \text{km/h} = 30.6\ \text{m/s}$$

$$v_2 = 90\ \text{km/h} = 25\ \text{m/s}$$

$$F = bv_1^2 = mg\ \sin\ \theta$$

$$b = \frac{mg}{v_1^2} \sin\ \theta$$



On level ground,

$$P = Fv_2 = (bv_2^2)(v_2) = bv_2^3$$

$$= \left(\frac{mg\ \sin\ \theta}{v_1^2}\right)(v_2^3) = \frac{(1600\,\text{kg})(9.8\,\text{m/s}^2)(0.1)(25\,\text{m/s})^3}{(30.6\,\text{m/s})^2} = 2.67 \times 10^4\ \text{W}$$

$$= (2.67 \times 10^4\text{W})\left(\frac{1}{746}\ \frac{\text{hp}}{\text{W}}\right) = 35.8\,\text{hp}$$

7.4 SUMMARY OF KEY EQUATIONS

Work: $W = F\ \cos\theta$ $\quad W = \int_{x_1}^{x_2} F(x)\ dx$

Hooke's law: $F = -kx$

Work to stretch a spring: $W = \frac{1}{2}\ kx^2$

Kinetic energy: $KE = \frac{1}{2}\ mv^2$

Work-energy theorem: $W = \frac{1}{2}\ mv_2^2 - \frac{1}{2}\ mv_1^2$

Power: $P = \frac{W}{t}$ or $P = \frac{E}{t}$

In general: $P = \frac{dW}{dt}$ or $P = \frac{dE}{dt}$

Supplementary Problems

SP 7.1 A man pushes at constant speed a 50-kg refrigerator a distance of 14 m across a level floor where the coefficient of friction is 0.40. How much work does he do?

SP 7.2 A toy dart gun utilizes a spring with a spring constant of 60 N · m. How much work must be done to compress this spring a distance of 3.2 cm?

SP 7.3 People have survived falls from great heights, provided that they landed in snow or foliage or some other such material to cushion the impact. A Russian pilot survived a fall from 22,000 ft when he landed on a sloping snow bank. The chances of survival depend on how you are oriented on impact. If you land head first, your chances of survival are greatly reduced. A typical adult of mass 80 kg landing flat on his back has about a 50 percent chance of survival if the impact force does not exceed $1.20 \times 10^5 N$. The terminal speed with which a person falling from a great height will strike the ground is about 140 mi/hr, or 63.0 m/s. At this impact speed, what depth of snow would be required if the average stopping force is not to exceed that corresponding to 50 percent survival chance?

SP 7.4 The engine in a car traveling on a level road must overcome air resistance and must do work in deforming the tires as they roll (road resistance). At 70 km/h the effective drag forces due to these effects are approximately equal. The road resistance is essentially independent of speed, however, whereas the air resistance varies approximately as the square of the speed. (a) By what factor will the power delivered to the wheels increase if the speed is doubled? (b) By what factor will the number of miles per gallon be reduced if the speed is doubled?

SP 7.5 A rock of mass m is dropped from rest from a point a distance h above the ground. Use the work-energy theorem to calculate the speed of the rock when it hits the ground.

SP 7.6 A Jaguar car of mass $2M$ is racing a small Austin Healy sports car of mass M. Initially the Jaguar has half the kinetic energy of the Austin Healy, but when the Jaguar then speeds up by 10 m/s, the two vehicles have the same kinetic energy. What were the initial speeds of the two cars?

SP 7.7 Suppose that energy Q is required to accelerate a car from rest to speed v, neglecting friction. How much added energy would be required to increase the speed from v to $2v$?

SP 7.8 In 1997 the Hale-Bopp comet provided a brilliant spectacle in the night sky. It passed harmlessly by us and will not return for another 2400 years, but its presence raised the specter of another comet colliding with earth and causing extensive damage. It is speculated that a comet collision some 70 million years ago may have kicked up such a huge cloud of dust that it obliterated sunlight for several years and led to the extinction of the dinosaurs. It is estimated that the Hale-Bopp comet had a mass of about 2.7×10^{14} kg, and at its nearest approach was traveling about 63 km/s (about 140,000 mi/h). (a) What is the kinetic energy of such a comet? Express the answer in joules and in "megatons of TNT." The detonation of 1 million tons of TNT releases 4.2×10^{15} J of energy. (b) The detonation of 1 megaton of TNT will produce a crater of about 1 km diameter. The diameter of the crater is proportional to the one-third power of the energy released. What size crater would you expect Hale-Bopp to produce?

SP 7.9 An engineer is asked to design a crash barrier for runaway trucks that get out of control descending a steep grade near Lewiston, Idaho. The specifications call for stopping a truck of mass 25,000 kg moving at 24 m/s with a stopping acceleration not to exceed $5.0g$ (where $g = 9.8$ m/s^2). (a) What spring constant is required? (b) How much will the spring have to compress? Does the design sound feasible to you?

SP 7.10 An airplane experiences a drag force av^2 due to air passing over its surface. There is an additional *induced drag force* b/v^2 that results from the fact that the wings cause air to be pushed downward and slightly forward. From Newton's third law we see that the air will thus push back, exerting an upward lift force and a backward "induced drag" force. Thus the total drag force can be expressed as $F = av^2 + b/v^2$. At constant speed, the engine must provide a forward force that balances this drag force. For a small single-engine airplane typical values of a and b might be $a = 0.12\ \text{N}\cdot\text{s}^2/\text{m}^2$ and $b = 2.9 \times 10^7 \text{N}\cdot\text{m}^2/\text{s}^2$. Calculate the speeds, in terms of a and b, at which such a plane will have (a) the maximum horizontal range and (b) the maximum time in the air (maximum *endurance*).

Solutions to Supplementary Problems

SP 7.1 $F_f = \mu mg \qquad W = F_f x = \mu mg x \qquad W = (0.40)(50\text{kg})(9.8\ \text{m/s}^2)(14\ \text{m}) = 2740\ \text{J}$

SP 7.2 $W = \frac{1}{2}kx^2 = (0.5)(60\text{N}\cdot\text{m})(0.032\text{m})^2 = 0.03\ \text{J}$

SP 7.3 $W = Fd = \frac{1}{2}mv^2 \qquad d = \dfrac{mv^2}{2F} = \dfrac{(80\ \text{kg})(63.0\ \text{m/s})^2}{2(1.20 \times 10^5\ N)} = 1.32\ \text{m}$

SP 7.4 Let F_R = drag force due to road resistance and $F_A = bv^2$ = drag force due to air resistance.

(a) $P_1 = (F_R + F_A)v_1 = (F_R + bv_1^2)v_1 \qquad P_2 = (F_R + bv_2^2)v_2 = F_R + b(2v_1)^2(2v_1)$

$$\frac{P_2}{P_1} = \frac{2v_1(F_R + 4bv_1^2)}{(F_R + bv_1^2)v_1} \qquad \text{Given } F_R = bv_1^2, \qquad \frac{P_2}{P_1} = \frac{2(F_R + 4F_R)}{F_R + F_R} = 5$$

(b) Mileage varies as mi/gal = distance traveled/energy used = vt/Pt. Thus

$$(\text{mi/gal})_1 = \frac{v_1}{P_1} \qquad (\text{mi/gal})_2 = \frac{v_2}{P_2} \qquad \frac{(\text{mi/gal})_2}{(\text{mi/gal})_1} = \frac{v_2}{v_1}\cdot\frac{P_1}{P_2} = \frac{2v_1}{v_1}\cdot\frac{1}{5} = \frac{2}{5}$$

Thus mileage decreases to 40 percent of value at 70 km/h when the speed is doubled.

SP 7.5 The work done by the force of gravity is equal to the gain in kinetic energy of the rock. Thus $W = Fh = mgh = 1/2\, mv^2$, and $v = \sqrt{2gh}$.

SP 7.6 Initially $KE_J = 1/2\ KE_{\text{AH}}$, and $1/2\ (2M)v_J^2 = 1/2\ (1/2\ Mv_{\text{AH}}^2)$. After speeding up $1/2\,(2M)(v_J + 10)^2 = 1/2\ Mv_{\text{AH}}^2$. Divide these equations:

$$\frac{(v_J + 10)^2}{v_J^2} = \frac{2v_{\text{AH}}^2}{v_{\text{AH}}^2} \qquad (v_J + 10)^2 = 2v_J^2$$

$$v_J + 10 = \sqrt{2}v_J \qquad v_J = 24\ \text{m/s} \qquad v_{\text{AH}} = 48\ \text{m/s}$$

SP 7.7 Energy to go from 0 to v is $E_1 = Q = 1/2\, mv^2$. Energy to go from 0 to 2 v is $E_2 = 1/2\ (2v)^2 = 4(1/2 mv^2) = 4Q$. So added energy to go from 0 to $2v$ is $E_2 - E_1 = 3Q$.

SP 7.8 (a) $$KE = \frac{1}{2}\, mv^2 = (0.5)(2.7 \times 10^{14}\ \text{kg})(63 \times 10^3\ \text{m/s})^2 = 5.36 \times 10^{23}\ \text{J}$$

$$= (5.36 \times 10^{23}\ \text{J})\left(\frac{1}{4.2 \times 10^{15}}\ \frac{\text{megaton}}{\text{J}}\right) = 1.28 \times 10^8\ \text{megaton TNT}$$

(b) $$\frac{d_2}{d_1} = \left(\frac{E_2}{E_1}\right)^{1/3} = \left(\frac{1.28 \times 10^8\ \text{megaton}}{1\ \text{megaton}}\right)^{1/3} = 503\ \text{km}$$

Thus Hale-Bopp would probably blast a crater about 500 km in diameter.

SP 7.9 The kinetic energy of the truck will do the work necessary to compress the spring.

$$W = \frac{1}{2}\, kx^2 = \frac{1}{2}\, mv^2 \tag{i}$$

The acceleration is not to exceed $5g$, so

$$F = kx = ma = 5\,mg \tag{ii}$$

Substitute Eq. (*ii*) in Eq. (*i*):

$$\frac{1}{2}\,(5\,mg)x = \frac{1}{2}\, mv^2 \qquad x = \frac{v^2}{5g} = \frac{(24\ \text{m/s})^2}{5(9.8\ \text{m/s}^2)} = 11.8\,\text{m}$$

$$k = \frac{5\,mg}{x} = \frac{(5)(25{,}000\ \text{kg})(9.8\ \text{m/s}^2)}{11.8\,\text{m}} = 1.04 \times 10^5\ \text{N} \cdot \text{m}$$

This doesn't sound like a great idea to me. It would require a mighty big spring, and one would have to have a latch to keep the compressed spring from shooting the truck back up the hill. But then, what do I know? I never thought Xerox photocopiers and CD players would work either.

SP 7.10 (a) $$F = av^2 + \frac{b}{v^2} \qquad P = Fv = av^3 + \frac{b}{v}$$

Fuel energy used in time t is:

$$E = Pt = (av^3 + \frac{b}{v})t$$

Range in time t is $x = vt$, so

$$E = (av^3 + \frac{b}{v})(\frac{x}{v}) \qquad \text{and} \qquad x = \frac{Ev^2}{av^4 + b}$$

Maximize x by varying v, with E constant (total fuel):

$$\frac{dx}{dv} = 0 = \frac{2Ev}{av^4 + b} - \frac{Ev^2(4av^3)}{(av^4 + b)^2}$$

So $\quad 2v\,(av^4 + b) - 4av^5 = 0 \qquad v = \left(\frac{b}{a}\right)^{1/4}$ for maximum range

(b) From above,

$$E = (av^3 + \frac{b}{v})t$$

Vary v to maximize t for longest flight time,

$$t = \frac{Ev}{av^4 + b} \qquad \text{and} \qquad \frac{dt}{dv} = 0 = \frac{E}{av^4 + b} - \frac{4Eav^4}{(av^4 + b)^2}$$

$av^4 + b - 4av^4 = 0 \qquad v = \left(\frac{b}{3a}\right)^{1/4}$ for maximum time in air

Chapter 8

Potential Energy and Conservation of Energy

8.1 POTENTIAL ENERGY

Consider a hockey puck sliding across an ice rink. Because of its motion, it has kinetic energy. As it slides, it does work against the force of friction and steadily slows to a stop. When at rest, the puck has no kinetic energy. The kinetic energy of the puck has been lost to heat in doing work against friction, and we cannot get it back. Friction is an example of a **nonconservative force**. This means that the mechanical energy of an object or of a system is not conserved when friction forces are present. In physics we use conserved to mean "constant" or "not changing."

On the other hand, some forces (the force exerted by a stretched spring and the gravity force are important examples) are what I call "spring-back" forces. Most books call this kind of force a **conservative force.** Suppose, for example, you were to throw a ball straight up. When the ball leaves your hand, it has kinetic energy, but as it rises, the kinetic energy decreases until it is zero at the highest point. This kinetic energy is not "lost," as was the case when work was done against friction. The gravity force can pull the ball back down, allowing it to gain as much kinetic energy as was lost on the way up. If the ball were thrown up to rest on a window ledge, we might think of the lost kinetic energy as being stored, because at a later time we could push the ball off, allowing it to gain speed as it fell and thereby recover its lost kinetic energy. The moving ball could strike something on the ground and do work on it. The stored energy is called *potential energy* because it has the "potential" to do work. It is useful to describe this state of affairs by introducing the concept of **potential energy (PE)** $U(x,\ y,\ z)$. This is a scalar function associated with a conservative force. U depends only on the position of the object. If $W(A, B)$ is the work done on an object in moving it from point A to point B, then the potential energy function is defined such that

$$W(A,B) = U(A) - U(B) \tag{8.1}$$

The work done by gravity when an object of mass m moves from elevation y_1 to elevation y_2 is

$$W(y_1, y_2) = mg(y_1 - y_2) = U(y_1) - U(y_2) \tag{8.2}$$

This means the gravitational potential energy function $U(y)$ is of the form

$$U(y) = mgy + U_0 \tag{8.3}$$

But from Eq. 8.1, $W(x_1,\, x_2) = U(x_1) - U(x_2)$, so

$$U(x_1) - U(x_2) = \tfrac{1}{2}\, mv_2^2 - \tfrac{1}{2}\, mv_1^2$$

or

$$\boxed{U(x_1) + \tfrac{1}{2}mv_1^2 = U(x_2) + \tfrac{1}{2}mv_2^2} \tag{8.7}$$

Equation 8.7 is a remarkable result called the **law of conservation of energy**. It shows that the quantity $U + 1/2\ mv^2$ stays constant at all points along the trajectory of a particle acted on by a conservative force. This quantity is called the **total mechanical energy** E of the object.

$$E = U + \tfrac{1}{2}mv^2 = \text{constant} \tag{8.8}$$

or

$$\text{PE} + \text{KE} = E = \text{constant}$$

MEMORIZE Eq. 8.7 and be able to apply it to problems involving conservative forces. It is very important.

Problem 8.1 A rock of mass m is dropped from rest at a point a height h above the ground. Use the conservation of energy principle to determine the speed of the rock when it strikes the ground. Neglect friction.

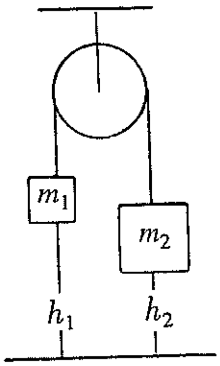
Solution

$$\text{PE}_1 + \text{KE}_1 = \text{PE}_2 + \text{KE}_2$$

$$mgh + 0 = 0 + \tfrac{1}{2}\, mv^2 \qquad v = \sqrt{2gh}$$

Here I chose ground level as the zero point of potential energy. The rock was initially at rest, so $KE_1 = 0$. Observe that this approach, using conservation of energy, is much simpler than the methods developed earlier.

Problem 8.2 An Atwood's machine consists of two masses m_1 and m_2 joined by a light cord which passes over a pulley. Initially the heavier mass is positioned a distance h above the floor. The masses are released from rest. At what speed are the masses moving when the heavier mass strikes the floor? Here $m_1 = 4$ kg, $m_2 = 6$ kg, and $h = 3$ m. The cord is long enough so that the lighter mass does not reach the pulley. Devices like this are used in the construction of elevators.



Solution $\text{PE}_1 + \text{KE}_1 = \text{PE}_2 + \text{KE}_2$

$$m_1gh_1 + m_2gh + 0 = m_1g\,(h_1 + h) + 0 + \tfrac{1}{2}(m_1 + m_2)v^2$$

$$v = \sqrt{2(\frac{m_2 - m_1}{m_2 + m_1})gh}$$

$$= \sqrt{2(\frac{6-4}{6+4})(9.8\,\mathrm{m/s^2})\,(3\,\mathrm{m})} = 3.4\,\mathrm{m/s}$$

Problem 8.3 Water flows over Niagara Falls at a rate of about 6000 $\mathrm{m^3/s}$, dropping a distance of 49 m. At what rate could electric power be generated if all of the potential energy loss of the water could be converted to electricity? One cubic meter of water has a mass of 1000 kg.

Solution $P = \frac{E}{t} = \frac{mgh}{t} = (\frac{m}{t})\,gh$

$$= (6000\frac{\mathrm{m^3}}{\mathrm{s}})(1000\frac{\mathrm{kg}}{\mathrm{m^3}})(9.8\frac{\mathrm{m}}{\mathrm{s^2}})(49\,\mathrm{m}) = 2.88 \times 10^9\ \mathrm{W} = 2.88\,\mathrm{GW}$$

By comparison, note that a nuclear power plant might generate about 1000 MW of power.

Problem 8.4 A skier passes over the crest of a small hill at a speed of 3.6 m/s. How fast will she be moving when she has dropped to a point 5.6 m lower than the crest of the hill? Neglect friction.

Solution

$$\mathrm{PE_1} + \mathrm{KE_1} = \mathrm{PE_2} + \mathrm{KE_2}$$

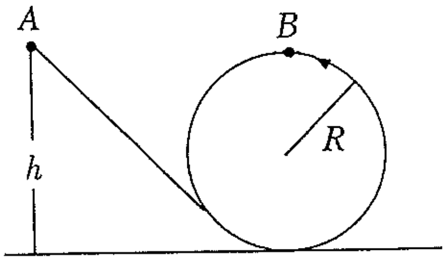
$$mgh_1 + \tfrac{1}{2}mv_1^2 = mgh_2 + \tfrac{1}{2}mv_2^2$$

$$v_2 = \sqrt{v_1^2 + 2g\,(h_1 - h_2)}$$

$$= \sqrt{(3.6\,\mathrm{m/s})^2 + 2\,(9.8\,\mathrm{m/s^2})\,(5.6\,\mathrm{m})}$$

$$= 11.1\,\mathrm{m/s}$$

Problem 8.5 In an amusement park roller coaster ride a car starts from rest at point A and races through a loop-the-loop. What is the minimum height h from which the car can start if it is not to leave the track at point B? The loop has radius R.



Solution If the car is just about to leave the track at point B, the normal force exerted on the car by the track at this point is zero. The only force acting on the car is then mg, and this must provide the needed centripetal force to keep the car moving along the circular track. Thus

$$mg = \frac{mv^2}{R}$$

We can find the speed v as a function of the starting elevation h by applying the conservation of energy principle.

$$mgh + 0 = mgR + \tfrac{1}{2}mv^2$$

Thus $\qquad v^2 = Rg \qquad gh = gR + \tfrac{1}{2}Rg \qquad h = \tfrac{3}{2}R$

8.2 ENERGY CONSERVATION AND FRICTION

The law of conservation of energy can be applied to systems where nonconservative forces like friction act. If a system does work against friction, the mechanical energy of the system will decrease. Thus if W_f is the work done against friction, then

Initial energy − energy lost to friction = final energy

$$E_1 - W_f = E_2$$

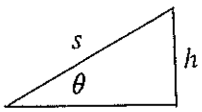
$$\boxed{U_1 + \tfrac{1}{2}mv_1^2 - W_f = U_2 + \tfrac{1}{2}mv_2^2} \tag{8.9}$$

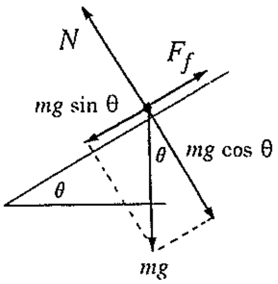
Problem 8.6 Near Lewiston, Idaho, is a steep grade heavily traveled by logging trucks. Several serious accidents have occurred when trucks lost their brakes and careened down the hill at high speed. Runaway truck ramps have been built that it is hoped can stop vehicles with no brakes. Suppose that a truck traveling 40 m/s encounters a ramp inclined up at 30° above horizontal. Loose gravel on the ramp provides a frictional force to help slow the truck as it moves up the ramp. The gravel has an effective coefficient of friction of 0.50. How far along the ramp would such a truck travel before coming to a stop?

Solution $\qquad N = mg \cos\theta$

$$F_f = \mu N$$

$$= \mu mg \cos\theta$$





$$U_1 + KE_1 - W_f = U_2 + KE_2$$

$$0 + \tfrac{1}{2}mv^2 - F_f s = mgh + 0 \qquad h = s\sin\theta$$

$$\tfrac{1}{2}mv^2 - (\mu mg\cos\theta)s = mgs\sin\theta$$

$$s = \frac{v^2}{2g(\sin\,\theta + \mu\,\cos\,\theta)} = \frac{(40\,\text{m/s})^2}{2(9.8\,\text{m/s}^2)(\sin\,30\,^\circ + 0.5\cos 30\,^\circ)}$$

$$= 87.5\,\text{m}$$

Problem 8.7 A package of mass m is dropped onto a conveyor belt moving at speed v. The coefficient of friction between the package and the belt is μ. (a) How far does the package move before it stops sliding on the belt? (b) How much work is done by the belt (including work against friction) before the package stops sliding?

Solution Initially the package has no horizontal velocity and the belt slides under it. Friction accelerates the package for a time t, until it reaches velocity v. Then the slipping stops and no more work is done against the friction force. During the slipping process, the package moves a distance x and the belt moves a distance x_B.

(a)

$$F_f = \mu mg = ma \qquad a = \mu g$$

$$v^2 = v_0^2 + 2\,ax = 0 + 2\,ax \qquad x = \frac{v^2}{2a} = \frac{v^2}{2\mu g}$$

$$v = v_0 + at = 0 + \mu gt \qquad t = \frac{v}{\mu g}$$

The package slides a distance $\Delta x = x_B - x$ on the belt, where

$$x_B = vt = v\left(\frac{v}{\mu g}\right) = \frac{v^2}{\mu g} \qquad \Delta x = \frac{v^2}{\mu g} - \frac{v^2}{2\mu g} = \frac{v^2}{2\mu g}$$

(b) The work done against friction is $W_f = F_f\,\Delta x$.

$$W_f = (\mu mg)\left(\frac{v^2}{2\mu g}\right) = \tfrac{1}{2}mv^2$$

The package gains kinetic energy KE = 1/2 mv^2. Thus the total work done by the belt is

$$W = W_f + \text{KE} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = mv^2$$

8.3 POTENTIAL ENERGY OF A SPRING

The force kx exerted by a spring is a conservative force. The work done in compressing or stretching a spring is stored as potential energy and can be used later to do work. We saw in Eq. 7.6 that the work done in stretching or compressing a spring of spring constant k by a distance x is 1/2 kx^2. Thus the potential energy of a spring is

$$\boxed{U = \tfrac{1}{2}kx^2} \qquad (8.10)$$

Here x is the displacement from the unstretched position of the end of the spring.

If a mass m is attached to the end of a spring and then allowed to oscillate back and forth, the energy of the system will remain constant.

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = E = \text{constant} \tag{8.11}$$

Problem 8.8 A mass m resting on a frictionless horizontal table is attached to the end of a spring of spring constant k. The other end of the spring is fixed. The mass is displaced a distance A from its equilibrium position and released from rest. What is the maximum speed of the mass as it oscillates?

Solution The total energy of the system is constant, so the kinetic energy will be greatest when the potential energy is a minimum, and this occurs when $x = 0$. Thus

$$E = U_1 + \mathrm{KE}_1 = U_2 + \mathrm{KE}_2$$

$$\tfrac{1}{2}kA^2 + 0 = 0 + \tfrac{1}{2}mv^2 \qquad v = \sqrt{\frac{k}{m}}\,A$$

Problem 8.9 An archery bow exerts a Hooke's law force kx on an arrow when the string is pulled back a distance x. Suppose that an archer exerts a force of 220 N in drawing back an arrow a distance of 64 cm. What is the spring constant of the bow? With what speed will an arrow of mass 24 g leave the bow?

Solution $k = \frac{F}{x} = \frac{220\,\mathrm{N{\cdot}m}}{0.64\ \mathrm{m}} = 344\,\mathrm{N\cdot m}$ $\qquad \mathrm{PE}_1 + \mathrm{KE}_1 = \mathrm{PE}_2 + \mathrm{KE}_2$

$$\tfrac{1}{2}kx^2 + 0 = 0 + \tfrac{1}{2}mv^2 \qquad v = \sqrt{\frac{k}{m}}\,x$$

$$x = \sqrt{\frac{344\,\mathrm{N{\cdot}m}}{0.024\,\mathrm{kg}}}\,(0.64\,\mathrm{m}) = 76.6\,\mathrm{m/s}$$

Problem 8.10 A crazy bungee cord jumper (there is no other kind) who weighs 800 N ties an elastic cord to his ankle and leaps off a high tower. The cord has an unstretched length of 30 m, and one end is attached to the point where the jumper starts. The effective spring constant of the elastic cord is 200 N · m. How far will the jumper fall before the cord stops his descent?

Solution Let the lowest point in the jump be $h = 0$. The initial kinetic energy and the kinetic energy at the lowest point are both zero, so energy conservation yields $mgh = 0 + 1/2\ kx^2$, where $x = h - 30$. Substitute $mg = 800$ N and $k = 200$ N/m, and solve.

$$h^2 - 68h + 900 = 0 \qquad h = 68 \pm \sqrt{(68)^2 - 4(900)} = 50\,\text{m}, \text{or } 18\,\text{m}$$

The correct solution is $h = 50\,\text{m}$. The solution $h = 18\,\text{m}$ corresponds to the jumper rebounding and compressing the bungee cord "spring," but a cord does not compress like a spring.

8.4 MACHINES

A simple machine is a device used to magnify a force or to change a small displacement into a large one. Common machines are a lever, an inclined plane, a block and tackle, a hydraulic jack, or a combination of gears. Typically work is done on the machine (the input work W_1), and then the machine in turn does some output work W_2. The energy state of the machine does not change appreciably during this process, so if friction is negligible, $W_1 = W_2$, based on the idea of energy conservation. Very often the input and output forces are constant, in which case $W_1 = W_2$ yields

$$F_1 d_1 = F_2 d_2 \qquad \text{or} \qquad \boxed{F_2 = \frac{d_1}{d_2} F_1} \tag{8.12}$$

Here F_1 acts over a distance d_1 and F_2 acts over a distance d_2. The **mechanical advantage** of the machine is defined as

$$\text{MA} = \frac{F_2}{F_1} \tag{8.13}$$

Problem 8.11 A pry bar is a device used to lift heavy objects (for example, a piano or large piece of machinery) a small distance, usually in order to place a wheeled dolly under the object. It consists of a long rod that rests on a fulcrum a short distance from the lifting end of the bar. Suppose the fulcrum of a pry bar is 3 cm from the load, and the point where you push down on the other end is 1.50 m from the fulcrum. What minimum force would you have to exert to lift a load of 2000 N? If you move the end of the bar down 4 cm, how much will you lift the load?

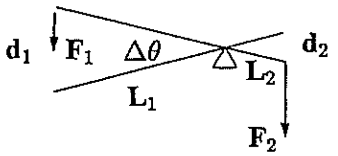
Solution If the bar rotates through a small angle $\Delta\theta$, then

$$d_1 = L_1\,\Delta\theta \quad \text{and} \quad d_2 = L_2\,\Delta\theta$$

$$F_1 L_1\,\Delta\theta = F_2 L_2\,\Delta\theta$$

$$F_1 = \frac{L_2}{L_1} \qquad F_2 = \left(\frac{0.03\,\text{m}}{1.50\,\text{m}}\right)(2000\,\text{N})$$

$$F_1 = 40\,\text{N}$$

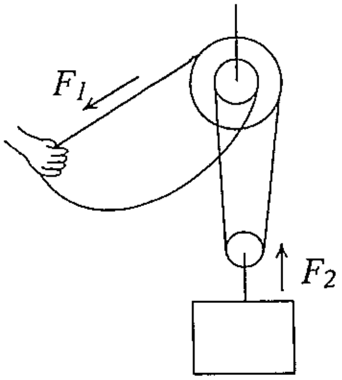


For similar triangles,

$$\frac{d_1}{d_2} = \frac{L_1}{L_2} \qquad d_2 = \frac{L_2}{L_1}d_1 = \frac{0.03\,\text{m}}{1.50\,\text{m}}\,(0.04\,\text{m}) = 0.008\text{m} = 8\text{ mm}$$

Note that a small input force results in a large output force, but the price one pays is that a large input displacement produces only a small output displacement.

Problem 8.12 Sketched here is a differential hoist of the kind used in a shop to lift an engine out of a car. The pulleys have teeth that mesh with a continuous chain. The top pulleys are welded together, and there are 18 teeth on the outer pulley and 16 teeth on the inner pulley. Thus when the pulley makes one revolution, 18 links of the chain are pulled up and 16 links are lowered, resulting in lifting the load. What is the mechanical advantage of this machine?



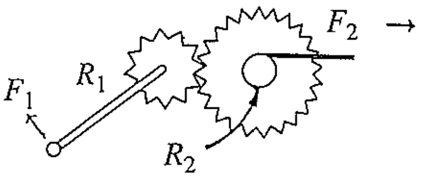
Solution Consider what happens when the top pulley makes 1 rev, that is, when the worker pulls 18 links of chain toward herself with force F_1. Let L = length of one link. The input work is thus $W_1 = 18\,LF_1$. The loop of chain that goes down to the load is thus shortened by 18 links and lengthened by 16 links, with a net shortening of $18L - 16L = 2L$. Shortening the loop by $2L$ lifts the load by L (try this with a piece of string to convince yourself of this tricky feature). Thus the output work is $W_2 = F_2L$. Neglecting friction,

$$W_1 = W_2 \qquad \text{or} \qquad 18LF_1 = F_2L$$

The mechanical advantage of the hoist is thus MA $= F_2/F_1 = 18$.

Problem 8.13 My sailboat trailer is equipped with a windlass that I use to pull my boat out of the water. It consists of a crank handle 30 cm long attached to the shaft of a small gear with 12 teeth. This small gear meshes with a larger gear with 36 teeth. Attached to this large gear is a drum of radius 2 cm on which is wound the line attached to the boat. (For you landlubbers, a line is a rope.) What tension can I apply to the line when I push on the crank with a force of 80 N?

Solution Consider what happens when the crank makes 1 rev. My hand moves a distance $d_1 = 2\pi R_1$. The large gear moves 12/36 = 1/3 rev. The line is thus pulled a distance $d_2 = 2\pi R_2/3$.



$$F_1d_1 = F_2d_2, \quad \text{so } F_2 = \frac{d_1}{d_2}F_1 = \frac{2\pi R_1}{2\pi R_{2/3}}F_1 = 3\frac{R_1}{R_2}F_1$$

$$F_2 = 3\left(\frac{30\text{ cm}}{2\text{ cm}}\right)(80\text{ N}) = 3600\,\text{N}$$

The mechanical advantage of the winch (neglecting friction) is 45. Amazing!

8.5 SUMMARY OF KEY EQUATIONS

If $U =$ potential energy: $F = -\frac{dU}{dx}$

Gravitational potential energy: $U = mgy$

Spring potential energy: $U = \frac{1}{2}kx^2$

Law of conservation of energy: $E = U + \frac{1}{2}mv^2 =$ constant

With friction present: $U_1 + \frac{1}{2}mv_1^2 - W_f = U_2 + \frac{1}{2}mv_2^2$
where $W_f = F_f d$

Machines: $W_1 = W_2$ or $F_1 d_1 = F_2 d_2$

Supplementary Problems

SP 8.1 Suppose that a rock of mass m thrown straight up will rise to a height h_1 in the absence of air drag. If a constant drag force of 0.1 mg acts, to what fraction of the height h_1 will the rock now rise? We have seen that in the absence of friction, the rise time is equal to the fall time. Is this still true if friction is present? Use energy conservation to reason this out.

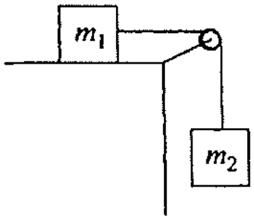
SP 8.2 A boy on a bridge throws a rock with speed v, and it lands in the water a distance h below. Calculate the speed with which the rock hits the water when it is thrown (a) horizontally, (b) at 45° above horizontal, and (c) straight down.

SP 8.3 Downhill ski racers always push off at the starting gate, assuming that so doing will give them a better time for the race. Calculate the speed of a racer after he has dropped 4.0 m in elevation below his starting point for the case of starting from rest and for starting with an initial speed of 1.0 m/s.

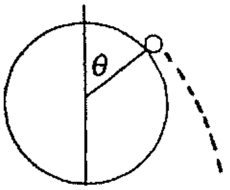
SP 8.4 Once many years ago I gave a lecture demonstration to illustrate the conservation of energy by means of the following setup. I tied a bowling ball to one end of a string, and I fastened the other end to the ceiling of the lecture hall. Holding the bowling ball while standing on a tall step ladder, I intended to show that when released from rest at the end of my nose, the ball would not swing higher and smash me in the face on the return swing. (Try this sometime if you want to experience a real game of "chicken." It's scary.) The demonstration made quite an impression on the class, but not for the reason I expected. Although the string was strong enough to hold the ball when it was motionless, when I let it go, the string broke at the bottom of the arc and the ball went bouncing around the room going "Boing, boing, boing" and scattering kids in every direction. Believe me, a bowling ball will really bounce on concrete. I'm sure they remember it to this day, long after they've forgotten about conservation of energy. Suppose the ball weighed 80 N and the string was 4.0 m long and had a breaking strength of 120 N. What is the maximum starting angle with vertical from which I could have released the ball without having the string break?

SP 8.5 You probably heard about the guy who fell off the top of the Empire State Building. At the 93rd floor a lady standing by an open window heard him say, "So far, so good. So far, so good." At the 54th floor they heard him say, "So far, so good. So far, so good." Sort'of like heading for the final exam in your physics class sometimes. Suppose someone had clocked this guy with a radar gun and found he was moving 12.0 m/s at one floor and 15.6 m/s one floor lower. Use energy principles to determine the distance between floors.

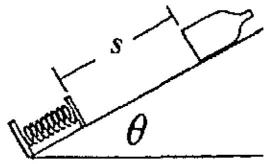
SP 8.6 Masses m_1 and m_2 are connected as shown here. Mass m_1 slides on a surface where the coefficient of friction is μ. Determine its speed after m_2 has fallen a small distance h.



SP 8.7 A small mass m is released from rest at the top of a frictionless spherical surface. At what angle with vertical will it leave contact with the sphere?



SP 8.8 In a bottling plant, containers travel along various conveyor belts between different units, such as sterilizers, fluid dispensers, capping machines, and labelers. At one point a bottle of mass m starts at rest and slides a distance s down a ramp inclined at angle θ above horizontal. There it strikes a spring of spring constant k. By how much does it compress the spring?

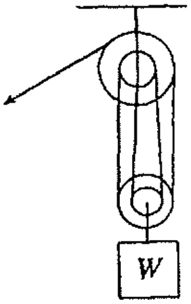


SP 8.9 A student designs a toy dart gun to propel a 12-g dart with a speed of 12.0 m/s by means of a spring that is to be compressed 2.00 cm to launch the dart. What spring constant is required?

SP 8.10 A 60- kg student finds she can run up a flight of stairs in a football stadium in 12 s. The flight has 120 steps, each 20 cm high. At what power level is she doing work? If her muscles are 20 percent efficient, how much energy does she use in this exercise? Express the result in kilocalories, remembering one Big Mac hamburger yields about 500 kcal. How much weight could you lose this way? To burn off 1 lb of fat, you must use about 3500 kcal.

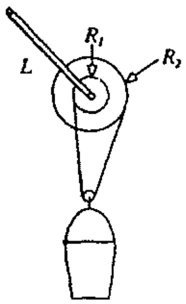
SP 8.11 A workman pushes a packing crate of weight W a distance s up a plane ramp inclined at angle θ above horizontal. How much work does he do? He pushes parallel to the plane. What force must he exert? What is the mechanical advantage of this simple machine?

SP 8.12 A woman uses the block and tackle shown here to lift a heavy weight W. Use energy principles to determine the mechanical advantage of this simple machine.



SP 8.13 A rocket of mass m is moving straight up with constant acceleration. At one point its velocity is v, and at an elevation h higher its velocity is $1.4v$. How much work was done by the rocket engine during this period?

SP 8.14 A Chinese windlass, of the kind used to lift water from wells in ancient times, is constructed with two cylinders of radii R_1 and R_2 mounted on a horizontal shaft. The shaft is turned



by a crank handle of length L. Rope is wound on the cylinders as shown here, and the hanging loop of rope lifts a bucket of weight W. What is the mechanical advantage of this machine?

Solutions to Supplementary Problems

SP 8.1 With no friction: $0 + \frac{1}{2}mv^2 = mgh_1 + 0 \qquad h_1 = \sqrt{\frac{v^2}{2g}}$

With friction: $\frac{1}{2}mv^2 - (0.1\,mg)\,h_2 = mgh_2 \qquad h_2 = 0.91h_1$

In the case where friction is present, consider two points A and B on the trajectory, both at the same elevation. Point A is reached on the way up, and point B is reached on the way down. Since both points are at the same elevation, the potential energy of the rock is the same at both points. However, since work was done against friction in going from A to B, the kinetic energy at B must be less than at A, so the rock is moving more slowly at B than at A. This is true for any two points at the same elevation along the trajectory, and hence the time required to fall will be greater than the rise time when friction is present. A more difficult problem is that of determining how friction affects the *total* time in the air. I leave that for you. It makes my head hurt.

SP 8.2 $0 + mgh = \frac{1}{2}mv^2 + 0, \qquad \text{so } v = \sqrt{2gh}$

The angle at which the rock is thrown doesn't enter into the result, so the speed at the water is independent of the angle at which the rock is thrown. This is counterintuitive for many people since they imagine that throwing the rock straight down will somehow give it greater speed at the ground. Not so. Note, however, that the horizontal component of the velocity will differ in the three cases, as will the vertical component of velocity.

SP 8.3 From rest: $0 + mgh = \frac{1}{2}mv_1^2 + 0 \qquad v_1 = \sqrt{2gh} = \sqrt{2(9.8\text{ m/s}^2)(4\text{ m})} = 8.85\text{ m/s}$

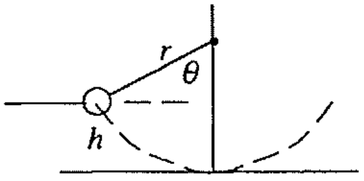
With $v_0 = 1$ m/s, $1/2\,mv_0^2 + mgh = 1/2\,mv_2^2 + 0$, so $v_2 = 8.91$ m/s. Note that $v_2 \neq v_1 + 1$!!!

SP 8.4 The string must provide enough upward force to balance the weight plus provide the radial force mv^2/r needed to make the ball curve upward. The tension in the string will thus be greatest at the lowest point in the arc, where the gravity force is directed straight down and the ball is moving fastest.

$$T - mg = \frac{mv^2}{r} \qquad \text{and} \qquad mgh = \frac{1}{2}mv^2$$

$$h = r - r\cos\theta \qquad T - mg = 2mg(1 - \cos\theta)$$

$$\cos\theta = 1 - \frac{T - mg}{2mg} = 1 - \frac{120 - 80}{(2)(80)} \qquad \theta = 41.4°$$



SP 8.5 $mgh_1 + \frac{1}{2}mv_1^2 = \frac{1}{2}mgh_2 + \frac{1}{2}mv_2^2 \qquad h_1 - h_2 = \frac{1}{2}(v_2^2 - v_1^2) = 5.07\text{ m}$

SP 8.6 Take $h = 0$ as the initial position of m_2. Then $\text{KE}_1 + \text{PE}_1 - W_f = \text{KE}_2 + \text{PE}_2$.

$$0 + 0 - \mu m_1 gh = \frac{1}{2}(m_1 + m_2)v^2 + m_2 g(-h) \qquad v = \sqrt{\frac{2(m_2 + \mu m_1)gh}{m_1 + m_2}}$$

SP 8.7 $mg\cos\theta = \frac{mv^2}{r} \qquad mgr + 0 = mgh + \frac{1}{2}mv^2 \qquad h = r\cos\theta$

$$\cos\theta = \frac{v^2}{rg} = 2 - 2\cos\theta \qquad \cos\theta = \frac{2}{3} \qquad \theta = 48.2\,°$$

SP 8.8 Let $h = 0$ at the bottle's lowest point and $h = (s + x)\sin\theta$ at the bottle's initial position. Here x = distance spring is compressed. Thus

$$mgh + 0 = 0 + \tfrac{1}{2}kx^2 + 0 \qquad x^2 - \frac{2mg}{k}(s + x)\sin\theta = 0$$

Let
$$A = \frac{2mg\sin\theta}{k}, \qquad \text{so } x^2 - Ax - As = 0$$

$$x = \frac{-A \pm \sqrt{A^2 + 4As}}{2} = 0.486\ \text{m} \qquad \text{or} \qquad -0.434\ \text{m}$$

The spring is thus compressed by 0.486 m. The solution –0.434 m corresponds to the spring when it is stretched out.

SP 8.9 $$\tfrac{1}{2}kx^2 = \tfrac{1}{2}mv^2 \qquad k = m\left(\tfrac{v}{x}\right)^2 = (0.012\ \text{kg})\left(\tfrac{12\ \text{m/s}}{0.02\ \text{m}}\right)^2 = 4320\ \text{N}\cdot\text{m}$$

SP 8.10 $$P = \frac{W}{t} = \frac{mgh}{t} = \frac{(60\ \text{kg})(9.8\ \text{m/s}^2)(120)(0.20\ \text{m})}{12\ \text{s}} = 1176W$$

$$\text{Work} = Pt = (1176W)(12\ \text{s}) = 1.41 \times 10^4\ \text{J}$$

$$\text{Work} = (1.41 \times 10^4\ \text{J})\left(\tfrac{1\ \text{kcal}}{4186\ \text{J}}\right) = 3.37\ \text{kcal}$$

If energy E is used, $0.2E$ = work or $E = 5$ work or 16.9 kcal.

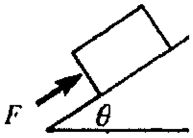
$$\text{Weight loss} = \frac{16.9\ \text{kcal}}{3500\ \text{kcal/lb}} = 0.005\ \text{lb}$$

It is tough to lose weight by exercising. It is better to eat less and reduce your calorie intake. Exercise is beneficial for other reasons.

SP 8.11 The crate is lifted vertically a distance $h = s\sin\theta$, so

$$\text{Work} = Fs = Wh = Ws\sin\theta$$

and
$$F = W\sin\theta \qquad \text{MA} = \frac{W}{F} = \frac{1}{\sin\theta}$$



SP 8.12 When the rope is pulled a distance d, the load is lifted a distance $1/4\ d$, so

$$Fd = W\left(\tfrac{d}{4}\right) \qquad \text{MA} = \frac{W}{F} = 4$$

SP 8.13 Work done by engine = gain in energy:

$$W = Fh = mgh + \tfrac{1}{2}m(1.4v)^2 - \tfrac{1}{2}mv^2 \qquad W = m(gh + 0.48v^2)$$

SP 8.14 Input work for 1 rev is $W_1 = 2\pi LF$. The rope is pulled up a distance $d_1 = 2\pi R_2$ and lowered a distance $2\pi R_1$. The loop of rope is shortened by $2\pi R_2 - 2\pi R_1$, so the load is lifted by half this amount.

$$d_2 = \tfrac{1}{2}(2\pi R_2 - 2\pi R_1) = \pi(R_2 - R_1) \qquad \text{MA} = \frac{d_1}{d_2} = \frac{2\pi R_2}{\pi(R_2 - R_1)} = \frac{2R_2}{R_2 - R_1}$$

Chapter 9

Linear Momentum and Collisions

A linebacker tries to tackle a big running back and is knocked over. A hailstorm flattens a wheat field in northern Idaho. Ninety cars pile up on a fog-covered freeway in southern California. Energetic subatomic particles cause radiation damage that incapacitates an electronic component in a space probe. All of these events involve collisions, and it turns out that they can most readily be understood in terms of linear momentum. This important concept has far-reaching philosophical consequences as well, and its conservation is related to our ideas about the homogeneity of space throughout the universe. I won't go into this aspect further, but perhaps this introductory treatment will whet your appetite to delve deeper into physics.

9.1 LINEAR MOMENTUM

The linear momentum **p** of a particle of mass m with velocity **v** is a vector quantity defined as

$$\boxed{\mathbf{p} = m\mathbf{v}} \tag{9.1}$$

Usually I refer to linear momentum simply as "momentum," with the understanding that by this I mean "linear momentum." Also, when dealing with one-dimensional motion, I do not use vector notation and simply refer to momentum as being positive or negative, as is done with velocity.

The most general formulation of Newton's second law of motion is

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}} \tag{9.2}$$

For a system for which the mass is constant,

$$F = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

This is the result stated previously. For systems where mass is not constant (for example, a rocket ejecting exhaust gases), Eq. 9.2 must be used.

Kinetic energy (KE) can be expressed in terms of momentum. If the magnitude of the momentum is $p = mv$, then

$$\boxed{KE = \tfrac{1}{2}mv^2 = \frac{p^2}{2m}} \qquad (9.3)$$

An interesting conclusion can be drawn concerning the behavior of two particles isolated from the outside world. This never happens in reality, but often it is a good approximation. Suppose the momenta of the particles are $\mathbf{p}_1$ and $\mathbf{p}_2$. If particle 1 exerts force $\mathbf{F}_{21}$ on particle 2 and particle 2 exerts force $\mathbf{F}_{12}$ on particle 1, then according to Newton's third law,

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \quad \text{or} \quad \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

Since

$$\mathbf{F}_{12} = \frac{d\mathbf{p}_1}{dt} \quad \text{and} \quad \mathbf{F}_{21} = \frac{d\mathbf{p}_2}{dt}$$

then

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

The total momentum of the system is $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, and the time rate of change of $\mathbf{p}$ is zero; thus

$$\boxed{\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}} \qquad (9.4)$$

Equation 9.4 is the **law of conservation of linear momentum** applied to an isolated system of two particles. Further, one can reason that the same law will apply to an isolated system of many particles as well.

Another way of expressing Eq. 9.4 is to say that the momentum of the system at one time is equal to the momentum of the system at a later time.

$$(\mathbf{p}_1 + \mathbf{p}_2)_{\text{init}} = (\mathbf{p}_1 + \mathbf{p}_2)_{\text{final}} \qquad (9.5)$$

If the vector momentum $\mathbf{p}$ is constant, then each of its x, y, and z components must also be constant.

Problem 9.1 A truck of mass 3000 kg traveling 5 m/s strikes a sedan stopped at a signal light. The two vehicles stick together. If the mass of the sedan is 2000 kg, at what speed does it move immediately after the collision?

Solution $P_{\text{before}} = P_{\text{after}} \qquad m_1v_1 + m_2v_2 = (m_1 + m_2)V$

$$(3000\,\text{kg})(5\,\text{m/s}) + 0 = (3000\,\text{kg} + 2000\,\text{kg})V \qquad V = 3.57\,\text{m/s}$$

Problem 9.2 A cannon of mass 1200 kg fires a 64-kg shell with a muzzle velocity of 62 m/s (this is the speed of the shell with respect to the cannon). Immediately after firing, what is the velocity V of the cannon and the velocity v of the shell with respect to the earth?

Solution The initial momentum of the system is zero. Just after firing, the cannon has velocity V (V is negative, since the cannon recoils to the left) and the shell moves to the right with velocity v with respect to the earth. The muzzle velocity is 62 m/s $= v - V$:

$$0 = m_1V + m_2v \qquad 0 = m_1V + m_2(V + 62)$$

$$0 = 1200V + 64(V + 62) \qquad V = -3.14\,\text{m/s} \qquad v = 58.9\,\text{m/s}$$

9.2 IMPULSE

Since $\mathbf{F} = d\mathbf{p}/dt$, $d\mathbf{p} = \int \mathbf{F}\, dt$ and the change in momentum $\Delta\mathbf{p}$ over a time interval t_1 to t_2 is

$$\boxed{\Delta\mathbf{p} = \int_{t_1}^{t_2} \mathbf{F}\, dt = \bar{\mathbf{F}}\ \Delta t} \tag{9.6}$$

Here $\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_2$ and $\Delta t = t_2 - t_1$ and $\bar{\mathbf{F}}$ is the average force that acts during the time interval Δt. $\Delta\mathbf{p}$ is called the **impulse** of the force **F**.

Note the following important observation. A moving particle has momentum and kinetic energy, but it *does not* carry with it a force, contrary to what many people imagine. The force required to cause a particle to stop (that is, to reduce its momentum to zero) depends on how big the momentum change is and on *how quickly* the momentum change occurs. A long collision time results in a smaller force, and a short collision time results in a larger force. If your head hits the hard surface of a car dashboard and stops quickly, a large (and possibly fatal) force will be applied to your head. If your head hits a cushioning air bag and stops slowly, the force will be greatly reduced. There are many examples of the consequences of this. When a parachutist lands, he bends his knees to lengthen the collision time with the earth, thereby reducing the force he experiences. A boxer rolls with the punch to minimize its effect. When you catch a hardball barehanded, you draw your hand back to reduce the sting.

Problem 9.3 High-speed photography reveals that when a bat strikes a baseball, a typical collision time is about 2 ms. If a speed of 45 m/s is imparted to a ball of mass 0.145 kg, what average force is exerted by the bat?

Solution $\bar{F}\,\Delta t = \Delta p = m\,\Delta v \qquad \bar{F} = \dfrac{(0.145\text{ kg})(45\text{ m/s})}{0.002\,\text{s}} = 3260\text{ N}$

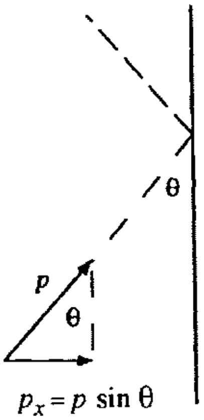
Problem 9.4 A ball of mass m and velocity v strikes a wall at an angle θ and bounces off at the same speed and at the same angle. If the collision time with the wall is Δt, what is the average force exerted on the wall?

Solution The component of the momentum perpendicular to the wall is $mv \sin\theta$ just perpendicular to the wall is $mv\sin\theta$ just before the collision and $-mv\sin\theta$ just after the collision. This means the change in momentum (the impulse) is

$$\Delta p_x = mv\sin\theta - (-mv\sin\theta)$$

$$= 2mv\sin\theta$$

Thus $$\bar{F} = \frac{\Delta p}{\Delta t} = \frac{2mv\sin\theta}{\Delta t}$$



Note that when the ball bounces back (as opposed to simply sticking to the wall), the force is *increased.*

Problem 9.5 Firefighters sometimes use a high-pressure fire hose to knock down the door of a burning building. Suppose such a hose delivers 22 kg of water per second at a velocity of 16 m/s. Assuming the water hits and runs straight down to the ground (that is, it doesn't bounce back), what average force is exerted on the door?

Solution Consider what happens in 1 s; that is,

$$\Delta t = 1\text{ s} \qquad \bar{F} = \frac{\Delta p}{\Delta t} = \frac{m\,\Delta v}{\Delta t} = (22\,\text{kg/s})(16\,\text{m/s}) = 352\text{ N}$$

9.3 COLLISIONS IN ONE DIMENSION

When two particles collide, the forces they exert on each other are much larger than any external forces acting. Thus we may assume that external forces are negligible, with the consequence that the momentum of the system remains constant. This means that **a system's momentum just before a collision is the same as the momentum just after the collision.** In a collision some kinetic energy of the particles is converted to heat, sound, elastic distortion, and so on. Such collisions are called **inelastic collisions**. Sometimes the loss in kinetic energy is negligible (as when two billiard balls collide). Such collisions are called **elastic collisions**, and for them the kinetic energy of the system is conserved before and after the collision. Of course, the kinetic energy of an individual particle can change, but the combined kinetic energy of both of the particles remains the same.

When two objects stick together, the collision is perfectly inelastic. If the particles bounce apart, it is hard to say at a glance if the collision was elastic or inelastic. In the problems you will encounter here, you will have to be told if the collision was elastic or inelastic. If mass m_1 has velocity v_1 and mass m_2 has velocity v_2 just before a perfectly inelastic collision (the particles stick together), the velocity V just after the collision is determined by

$$m_1v_1 + m_2m_2 = (m_1 + m_2)V \quad \text{or} \quad V = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \tag{9.7}$$

If the collision is perfectly elastic, masses m_1 and m_2 can have different velocities V_1 and V_2 after the collision. Since the kinetic energy remains constant,

$$m_1v_1 + m_2v_2 = m_1V_1 + m_2V_2 \tag{9.8}$$

and

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 \tag{9.9}$$

These two equations can be solved for the two unknown final velocities, V_1 and V_2. Multiply Eq. 9.9 by 2 and rearrange:

$$m_1(\ v_1^2 - V_1^2) = m_2(v_2^2 - V_2^2)$$

Factor both sides of this equation:

$$m_1(v_1 - V_1)(v_1 + V_1) = m_2(v_2 - V_2)(v_2 + V_2) \tag{9.10}$$

Rearrange Eq. 9.8:

$$m_1(v_1 - V_1) = m_2(v_2 - V_2) \tag{9.11}$$

Divide Eq. 9.10 by Eq. 9.11:

$$v_1 + V_1 = v_2 + V_2 \quad \text{or} \quad v_1 - v_2 = -(V_1 - V_2) \tag{9.12}$$

$v_1 - v_2$ is the relative velocity of particle 1 with respect to particle 2 before the collison, and $V_1 - V_2$ is the same quantity after the collision. Thus Eq. 9.12 yields the interesting result that the relative speed of one particle with respect to the other does not change in a collision. The relative velocity of each particle changes direction in a collision, but the relative speed is constant.

We can solve Eq. 9.12 for V_2 and substitute it back into Eq. 9.8, thereby yielding one equation for V_1. The same thing can be done with V_1 to find V_2. After some algebraic labor, we find:

$$V_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2\,m_2}{m_1 + m_2}\right)v_2 \tag{9.13}$$

$$V_2 = \left(\frac{2\,m_1}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 \tag{9.14}$$

These equations yield interesting results for some simple special cases.

1. $\mathbf{m_1 = m_2}$ Here $V_1 = v_2$ and $V_2 = v_1$. The particles exchange speeds. This is approximately what happens when pool balls collide.

2. $\mathbf{m_2}$ initially at rest Equations 9.13 and 9.14 become

$$V_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 \tag{9.15}$$

$$V_2 = \left(\frac{2m_1}{m_1 + m_2}\right)v_1 \tag{9.16}$$

If $m_2 \gg m_1$ (like a golf ball hitting a brick wall), we see that $V_1 \simeq -v_1$ and $V_2 \simeq v_2 = 0$. The big object remains at rest, and the small one bounces back with its speed unchanged.

If $m_1 \gg m_2$ (a locomotive hitting your parked motorcycle), then $V_1 = v_1$ and $V_2 = 2v_1$. The incident big particle continues with no change in speed, and the small stationary particle takes off with twice the speed of the incoming particle.

Problem 9.6 Here's an entertaining lecture demonstration. Place a small ball bearing of mass m_2 on top of a larger superball of mass m_1 ($m_1 \gg m_2$). Hold the two together at shoulder height (call this height h above the floor), and drop the two simultaneously onto a concrete floor. The result is impressive. The steel ball bearing takes off like a bat out of hell. The first time I did this, I broke the overhead fluorescent lights. Calculate the height to which the ball bearing would rise (if it doesn't hit the lights on the ceiling) if the two are dropped from height h. Assume all collisions are elastic. (*Hint*: Imagine that first the superball collides elastically with the floor, and then when it rebounds, it meets the falling ball bearing that is right behind it.)

Solution The superball hits the floor with speed v_1 where conservation of energy during the fall yields

$$0 + m_1gh = \tfrac{1}{2}\, m_1v_1^2 + 0 \qquad v_1 = \sqrt{\frac{2g}{h}}$$

The superball bounces up with speed v_1 and collides with the ball bearing, whose velocity is $v_2 = -v_1$. After the collision the velocity of the ball bearing is given by Eq. 9.16.

$$V_2 = \left(\frac{2m_1}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 \qquad \text{where } m_1 \gg m_2$$

$$V_2 \cong 2v_1 - v_2 = 2v_1 - (-v_1) = 3v_1 = 3\sqrt{\frac{2g}{h}}$$

Applying conservation of energy to the rising ball bearing yields the height h' to which the ball bearing rises.

$$\tfrac{1}{2}\, m_2 V_2^2 + 0 = 0 + m_2 g h' \quad \text{or} \quad h' = \tfrac{1}{2g}\left(3\sqrt{\tfrac{2g}{h}}\right)^2 = 9h\,!!!$$

Maybe we could use this technique to launch space vehicles or to send raw materials to manufacturing plants on the moon. Or how about lifting water to elevated levees in the Himalayan mountains in Nepal? (Too late: People thought of this idea long ago. It's called a *hydraulic ram.*)

9.4 THE CENTER OF MASS

In trying to understand the behavior of a system of particles or of an extended object like a baseball bat, it is useful to introduce the concept of the **center of mass** (CM). The center of mass is what I would call the "balance point," that is, like the fulcrum of a teeter-totter. For two particles of equal mass, the center of mass lies midway between them on the line joining them. For an object like a brick, the center of mass is at the geometrical center.

If a system consists of particles of mass m_1 at position $\mathbf{r}_1$, m_2 at position $\mathbf{r}_2$, ... , and m_N at $\mathbf{r}_N$, the position of the center of mass is defined to be

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N}{m_1 + m_2 + \cdots + m_n} \tag{9.17}$$

or

$$\boxed{\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N}{M}} \tag{9.18}$$

Here M is the total mass of the system. The x, y, and z coordinates of the center of mass are

$$x = \frac{m_1x_1 + m_2\, x_2 + \cdots + m_N\, x_N}{M} \tag{9.19}$$

$$y = \frac{m_1y_1 + m_2\, y_2 + \cdots + m_N\, y_N}{M} \tag{9.20}$$

$$z = \frac{m_1z_1 + m_2\, z_2 + \cdots + m_N\, z_N}{M} \tag{9.21}$$

Observe that the total momentum of a system subject to no external forces is constant.

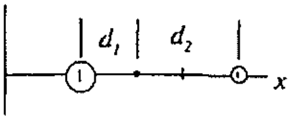
$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_N\mathbf{v}_N = \frac{d}{dt}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N)$$

$$= \frac{d}{dt}(M\mathbf{R}) = \text{a constant}$$

Thus we can think of the behavior of the system like this: Imagine all of the mass of the system concentrated at the center of mass point. If no external forces act, this center of mass point will then move with constant velocity. It can be seen also that if external forces act, the center of mass moves as if it were a particle of mass M subject to those forces. For example, consider an artillery shell moving through the air along a parabolic path. If the shell explodes in midair, pieces will fly off in all directions, but the center of mass point will continue moving along the parabolic path as if nothing had happened.

Problem 9.7 Two masses are placed on the x axis. 4 kg is at $x = 1$ m, and 2 kg is at $x = 4$ m. What is the position of the center of mass?

Solution $x = \dfrac{(4\,\text{kg})(1\,\text{m}) + (2\,\text{kg})(4\,\text{m})}{4\,\text{kg} + 2\,\text{kg}} = 2\,\text{m}$



We see that the center of mass is closer to the larger mass. The distance from each mass to the center of mass point is in the inverse ratio of the masses; that is, $d_1/d_2 = m_2/m_1$. Here $m_1 = 4\ kg$, $m_2 = 2$ kg, $d_1 = 1\ m$, and $d_2 = 2\ m$.

We can find the center of mass of a continuous mass distribution by breaking the object into many little pieces, each of mass dm and volume dV. The mass density ρ of a material is defined as the mass per unit volume, $\rho = dm/dV$. If a small-volume element is located at $(x,\ y,\ z)$, the coordinates of the center of mass become

$$x = \frac{1}{M}\int x dm = \frac{1}{M}\int \rho x dV \qquad (9.22)$$

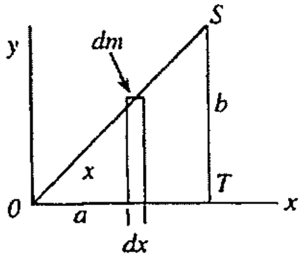
$$y = \frac{1}{M}\int y dm = \frac{1}{M}\int \rho y dV \qquad (9.23)$$

$$z = \frac{1}{M}\int z dm = \frac{1}{M}\int \rho z dV \qquad (9.24)$$

For symmetric objects of uniform density, the center of mass is at the geometric center. Note that the center of mass does not have to be within the object.

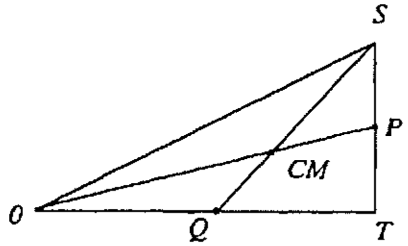
Problem 9.8 A uniform sheet of metal is cut into a right triangle. Its surface mass density (in kilograms per square meter) is σ. Determine the position of the center of mass.

Solution To find the x coordinate of the CM, break the triangle into narrow strips parallel to the y axis. The mass of a strip is $dm = \sigma\, y dx$, where ydx is the area of the strip. Note that $y/x = b/a$, so $dm = \sigma bxdx/a$. Equation 9.23 thus yields



$$x = \frac{1}{M}\int_0^a \frac{\sigma b x^2}{a}\, da = \frac{\sigma b a^3}{3Ma} \qquad M = \tfrac{1}{2}ab\sigma, \text{ so } x = \tfrac{2}{3}a$$

A similar calculation can be carried out to find Y. However, we need not go to the work of doing this. Rather, notice that the x coordinate of the CM is two-thirds of the way from the vertex at O, so the y coordinate of the CM is two-thirds of the way from vertex S, at $Y = b/3$.

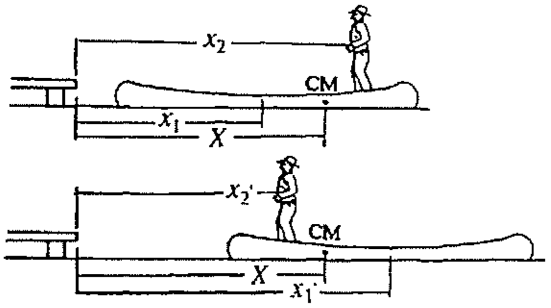


An ingenious way of solving this problem without using integrals is the following: Observe that the CM of each little strip must lie at its midpoint. Thus the CM of the triangle must lie somewhere on the line OP, where point P is the midpoint of side ST. By similar reasoning, if we break the triangle into strips parallel to the x axis, we see that the CM of each strip must lie at its midpoint, and so the CM of the triangle must lie along line QS. Thus the CM of the triangle lies at the point where OP and QS intersect. The location of this point is found using geometry. Remember, THINK before you start calculating madly. Elegance is the essence of mathematics and physics!

Problem 9.9 Maybe you've had an experience like this. One day while canoeing with my grandkids, I tried to climb out of the canoe onto the dock. As I stepped out of the canoe, it moved off in the opposite direction, dumping me in the lake. Great screams of laughter all around. Suppose I (of mass 90 kg) started 2 m from the midpoint of the canoe (mass 30 kg) and walked 4 m along the canoe toward the opposite end. How far through the water would the canoe move?

Solution Assuming no net external force acts on the system, the CM will not move. The CM of a uniform canoe is at its center. Take $x = 0$ at the dock. The CM of the system is a distance X from the dock. Initially

$$X = \frac{m_1 x_1 + m_2 x_2}{M}$$



After moving,

$$X' = \frac{m_1 x_1' + m_2 x_2'}{M}$$

$X' = X$, since the CM doesn't move,

$$m_1 x_1 + m_2 x_2 = m_1 x_1' + m_2 x_2'$$

$$x_2 = x_1 + 2, \qquad x_2' = x_1' - 2$$

The canoe moves a distance d of $x'_1 - x_1$:

$$m_1 x_1 + m_2(x_1 + 2) = m_1 x'_1 + m_2(x'_1 - 2)$$

Solve with $m_1 = 30\,\text{kg}$, and $m_2 = 90\,\text{kg}$. Find $d = x'_1 - x_1 = 3\,\text{m}$.

Note that once you conclude that the CM is 0.5 m from the person, you could arrive at the answer by looking at the drawing. The distance between the person and the midpoint of the canoe is 2.0 m, so the CM must be 0.5 m from the person and 1.5 m from the canoe midpoint (a ratio of 1:3) since the masses 30 kg and 90 kg are in this ratio.

9.5 ROCKETS

Consider a rocket plus fuel that is initially at rest. When fuel is ejected out the back of the rocket, it acquires momentum, and so the rocket must move forward to acquire opposite momentum to cancel the fuel's momentum, since the total momentum of the system remains constant. Suppose that a rocket of mass M is moving at speed v with respect to the earth. Now a mass of fuel Δm is ejected with speed v_e with respect to the rocket. This means the fuel is moving with velocity $v - v_e$ with respect to the earth, and the rocket now moves forward with mass $(M - \Delta m)$ and velocity $v + \Delta v$.

Momentum conservation requires that $Mv = (M - \Delta m)\,(v + \Delta v) + \Delta m\;(v - v_e)$. Simplify $M\Delta v = \Delta m\,(v_e)$. The change in mass of the rocket is $M = -\Delta m$. In the limit $\Delta v \to dv$ and $\Delta m \to dm$, we obtain $Mdv = \;-v_e\,dM$. Integrate:

$$\int_{v_i}^{v_f} dv = \;-v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \tag{9.25}$$

The force propelling the rocket is called the *thrust*.

$$\text{Thrust} = F = M\frac{dv}{dt} = -v_e\frac{dM}{dt} \tag{9.26}$$

$$\frac{dM}{dt} < 0, \qquad \text{so the thrust is positive (forward)}$$

Problem 9.10 The Saturn V rocket had a mass of 2.45×10^6 kg, 65 percent of which was fuel. In the absence of gravity and starting at rest, what would be the maximum velocity attained (the "burnout velocity")? The fuel exhaust velocity was 3100 m/s.

Solution From Eq. 9.25,

$$v - 0 = v_e \ln\frac{M_i}{M_f} = 3100 \ln\frac{M}{0.35\,M} = 3250\,\text{m/s}$$

In the preceding discussion of rockets, I have not included the effect of gravity. If a rocket is fired straight up, the thrust of the rocket will be reduced by the weight mg, and this will reduce the final velocity in Eq. 9.25 by gt, where t is the burn time for the fuel.

9.6 SUMMARY OF KEY EQUATIONS

$$\mathbf{p} = m\mathbf{v}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\text{If } \mathbf{F}_{\text{ext}} = 0, \text{then } \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \cdots + = \text{constant}$$

$$\text{In } \textit{all} \text{ collisions } \mathbf{p}_{\text{before}} = \mathbf{p}_{\text{after}}$$

$$\text{In } \textit{elastic collisions } \text{KE}_{\text{before}} = \text{KE}_{\text{after}}$$

Impulse: $\Delta\mathbf{p} = \int \mathbf{F}\ dt = \bar{\mathbf{F}}\,\Delta t$

Elastic collision: If $v_1$and V_1 are initial and final velocities of m_1 and v_2 and V_2 are initial and final velocities of m_2, then:

m_1, v_1, m_2, v_2 $\quad V_1 = (\frac{m_1 - m_2}{m_1 + m_2})v_1 + (\frac{2\,m_2}{m_1 + m_2})\,v_2$

m_1, V_1, m_2, V_2 $\quad V_2 = (\frac{2\,m_1}{m_1 + m_2})v_1 + (\frac{m_1 - m_2}{m_1 + m_2})\,v_2$

Center of mass: $\mathbf{R} = \dfrac{m_1\,\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N}{m_1 + m_2 + \cdots + m_N}$

Rockets: $v_f - v_i = v \ln \dfrac{M_f}{M_i}$

Rocket thrust: $F = -v_e\dfrac{dM}{dt}$

Supplementary Problems

SP 9.1 Calculate the KE and momentum of a particle of mass 0.020 kg and speed 65 m/s.

SP 9.2 Once I tried to determine the speed of an arrow from my bow in the following way. I stuffed a big cardboard box full of newspapers (total mass 2.0 kg). I placed the box on a table where the coefficient of friction was 0.30. Then I fired my arrow horizontally into the box. The arrow mass was 0.030 kg. The arrow stuck in the box and caused the box to slide 24 cm. What was the speed of the arrow?

SP 9.3 If a soldier shoots an enemy with the intent of killing him, he wants the bullet to deliver as much energy as possible. On the other hand, if you simply want to knock someone down (as in riot control), you want to deliver the maximum force. To see that these are not the same considerations, calculate the energy transferred to a very massive wooden block and the average force exerted on the block when a bullet of mass 0.008 kg and velocity 400 m/s makes a collision of duration 6.0 ms. Consider the case of a rubber bullet that bounces back from the block with no loss in speed (a perfectly elastic collision) and the case of an aluminum bullet that sticks in the block (a perfectly inelastic collision).

SP 9.4 Marilyn Vos Savant writes a newspaper column in which she answers questions sent in by readers. Although her credentials list her as having recorded the highest IQ test score ever, she came up with the wrong answer to this stickler. Suppose you are driving your car at high speed and face two choices. You can hit a brick wall head on, or you can hit an oncoming car identical to yours and moving at the same speed as you. In both cases assume your car sticks to whatever it hits and that the collision times are the same in both cases. In truth, the collision time might be a little longer if you hit another car because modern cars are designed to be "crushable" in order to lengthen the collision time. Calculate the force experienced in each case. (Most people think it is better to hit the wall.)

SP 9.5 A football running back of mass 90 kg moving 5 m/s is tackled head on by a linebacker of mass 120 kg running 4 m/s. They stick together. Who knocks whom back, and how fast are they moving just after the tackle?

SP 9.6 A machine gun fires 4.8 bullets per second at a speed of 640 m/s. The mass of each bullet is 0.014 kg. What is the average recoil force experienced by the machine gun?

SP 9.7 Two astronauts, Joe and Katie, each of mass $2M$, are floating motionless in space. Joe throws a compressed air cylinder of mass M with speed v toward Katie. She catches it and throws it back with speed v (with respect to herself). Joe again catches the cylinder. What will be his speed after so doing?

SP 9.8 The cabin section of a spacecraft is separated from the engine section by detonating the explosive bolts that join them. The explosive charge provides an impulse of 400 N · s. The cabin has a mass of 1000 kg, and the engine compartment has a mass of 1400 kg. Determine the speed with which the two parts move apart.

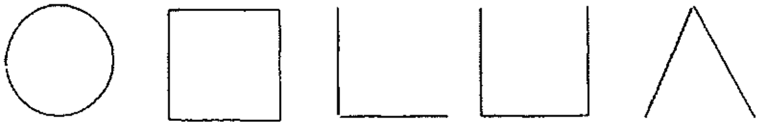
SP 9.9 Water impinges on a fixed turbine blade with velocity v. The blade is curved so that it deflects the water by 180° and directs it back in its initial direction with no loss in speed. The mass of water striking the blade per unit time is μ. What force is exerted on the blade?

SP 9.10 An object of mass m and velocity v collides elastically with a stationary object and continues in the same direction with speed $0.25v$. What is the mass of the object that was initially stationary?

SP 9.11 A rocket in deep outer space turns on its engine and ejects 1 percent of its mass per second with an ejection velocity of 2200 m/s. What is the initial acceleration of the rocket?

SP 9.12 A boy of mass 40 kg stands on a log of mass 60 kg. The boy walks along the log at 2 m/s. How fast does the log move with respect to the shore?

SP 9.13 The objects shown here are constructed by bending a uniform wire. Determine the approximate position of the center of mass for each using symmetry and graphical methods (as opposed to using equations).



Solutions to Supplementary Problems

SP 9.1

$$\mathrm{KE} = \tfrac{1}{2}mv^2 = (0.5)\,(0.020\,\mathrm{kg})\,(65\ \mathrm{m/s})^2 = 0.65\mathrm{J}$$

$$P = mv = (0.020\ \mathrm{kg})\,(65\ \mathrm{m/s}) = 1.3\ \mathrm{kg}$$

SP 9.2

$$mv = (m+M)V \qquad \tfrac{1}{2}\,(m+M)V^2 = F_f\,s = \mu\,(m+M)gs$$

$$v = \left(\frac{m+M}{m}\right)\sqrt{2\mu gs} = 80\mathrm{m/s}$$

SP 9.3 The rubber bullet has the same KE before and after the collision, so it transferred no energy to the block. The force it exerted on the block is

$$F_r \cong \frac{\Delta p}{\Delta t} = \frac{mv-(-mv)}{\Delta t} = \frac{2\,mv}{\Delta t} = \frac{(2)(0.008\,\mathrm{kg})(400\mathrm{m/s})}{\mathrm{s}} = 1070\,\mathrm{N}$$

For the aluminum bullet $mv = (m+M)V$. Loss in momentum of the bullet is

$$\Delta p = mv - mV = mv - \left(\frac{m^2}{m+M}\right)v$$

$$= \frac{mM}{m+M}v \simeq mv \qquad \text{since } M \gg m$$

$$F = \frac{\Delta p}{\Delta t} \simeq \frac{mv}{\Delta t} = 535N$$

Energy transferred to the block is

$$\Delta\,\mathrm{KE} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}\,mV^2 \simeq \tfrac{1}{2}\,mv^2 - \tfrac{1}{2}\,m\left(\frac{m}{m+M}\right)^2 v^2$$

$$\Delta\,\mathrm{KE} \simeq \tfrac{1}{2}mv^2 \qquad \text{if } M \gg m$$

Thus the rubber bullet exerts a greater force and is more likely to knock the block over, and the aluminum bullet transfers more energy to the block.

SP 9.4 If your car comes to a stop, $\Delta p = p - 0 = p = mv$. The force it experiences is

$$F \cong \frac{\Delta p}{\Delta t} = \frac{mv}{\Delta t}$$

Thus the force is the same whether you hit the wall or the oncoming car. Of course, if the other car had more momentum (for example, was larger or moving faster), you would be knocked back and you would experience a larger force. The linebacker knocks back the running back.

SP 9.5 $m_1v_1 + m_2v_2 = (m_1+m_2)V \quad V = \dfrac{m_1v_1+m_2v_2}{m_1+m_2} = \dfrac{(90\mathrm{kg})\,(5\mathrm{m/s}) + (120\,\mathrm{kg})(-4\,\mathrm{m/s})}{90\mathrm{kg}+120\mathrm{kg}} = -0.14\ \mathrm{m/s}$

SP 9.6

$$\bar{F} = \frac{\Delta p}{\Delta t\Delta} = \frac{mv}{\Delta t} = (4.8\mathrm{s}^{-1})\,(0.014\mathrm{kg})\,(640\mathrm{m/s}) = 43N$$

SP 9.7 Joe throws the cylinder: $0 = Mv_c + 2Mv_J$, and $v_c - v_J = v$, so

$$v_c = \frac{2v}{3} \qquad v_J = -\frac{v}{3}$$

Katie catches the cylinder:

$$M\left(\frac{2v}{3}\right) + 0 = (M+2M)v_{Kc} \qquad v_{Kc} = \tfrac{2}{9}v$$

Katie throws the cylinder:

$$(M+2M)(\tfrac{2v}{9}) = Mv_c{}' + 2Mv_K \qquad v_K - v_c' = v$$

so

$$\tfrac{2}{3}v = v_c' + 2(v + v_c') \qquad v_c' = -\tfrac{4}{9}v$$

Joe catches the cylinder:

$$(2M)(-\tfrac{v}{3}) + M(-\tfrac{4}{9}v) = (2M+M)v_J' \qquad v_J' = -\tfrac{10}{27}v$$

The final speed of Joe and the cylinder is $0.37v$. Check: The total momentum of the system must remain zero. Thus

$$(M+2M)v_J' + 2Mv_K = 0$$

$$(3M)(-\tfrac{10}{27}v) + 2M(\tfrac{5}{9}v) = 0$$

using

$$v_K = v + v_c' = v - \tfrac{4}{9}v = \tfrac{5}{9}v \qquad 0 = 0 \quad \text{OK}$$

SP 9.8

$$\Delta p = \bar{F}\Delta t = mv$$

Thus

$$v = \frac{\Delta p}{m} \qquad v_1 = \frac{400\,\text{N·s}}{1000\,\text{kg}} = 0.4\,\text{m/s}$$

$$v_2 = \frac{400\,\text{N·s}}{1400\,\text{kg}} = 0.286\,\text{m/s} \qquad v = v_1 + v_2 = 0.686\ \text{m/s}$$

SP 9.9

$$F \cong \frac{\Delta p}{\Delta t} = \frac{mv-(-mv)}{\Delta t} = \frac{2mv}{\Delta t} = 2\mu v$$

SP 9.10 From Eq. 9.15,

$$V_1 = \frac{m_1 - m_2}{m_1 + m_2}v_1 \qquad \frac{v}{4} = \frac{m - m_2}{m + m_2}v_1 \qquad m_2 = 0.6\text{m}$$

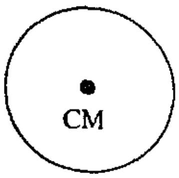
SP 9.11 From Eq. 9.26,

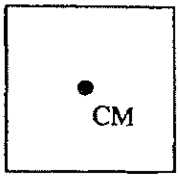
$$F = -v_e\frac{dM}{dt} = Ma \qquad a = -\frac{v_e}{M}\frac{dM}{dt} = -(2200\ \text{m/s})(-0.01) = 22\ \text{m/s}^2$$

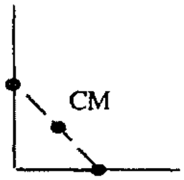
SP 9.12 The momentum of the system is zero: $m_Bv_B + m_Lv_L = 0$.

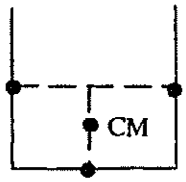
$$m_B(v - v_L) + m_Lv_L = 0 \qquad v_L = \frac{m_B}{m_B + m_L}v \qquad v_L = \left(\frac{40}{40+60}\right)(2\ \text{m/s}) = 0.8\ \text{m/s}$$

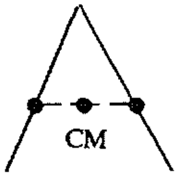
SP 9.13











The CM for each straight section is at its midpoint. I treat each of them as a point mass. For the U shape I found the CM for the two parallel sides and weighted this as worth $2M$. I then combined this with M for the bottom section.

Chapter 10

Rotational Motion

Rotational motion plays an important role in nature, and here we investigate the behavior of rigid bodies when they rotate. A rigid body is one that does not deform as it moves. The equations involved here are similar to those that describe linear translational motion.

10.1 ANGULAR VARIABLES

Consider a planar object rotating about an axis perpendicular to its plane. We describe the position of a point on the object by the coordinates r and θ, where θ is measured with respect to the x axis, as in Figure 10.1. When the object turns through an angle θ, the point moves a distance s along the arc. We define the angle θ in **radians** as

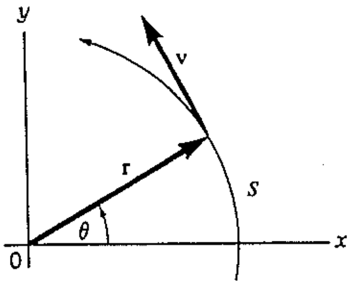


Figure 10.1

$$\theta = \frac{s}{r} \qquad \text{or} \qquad \boxed{s = r\theta} \qquad (10.1)$$

You can see that if θ is doubled, the arc length s will also be doubled. Since θ is the ratio of two lengths, it is a dimensionless quantity. The circumference of a circle is $s = 2\pi r$ so θ for a full circle is 2π. Thus $2\pi\,\text{rad} = 360°$. It is easy to convert radians to degrees or degrees to radians using a ratio.

$$\frac{\theta\,(\text{radians})}{\theta\,(\text{degrees})} = \frac{2\pi}{360°}$$

Exercise 10.1 Express 45°, 60°, 90°, and 170° in radians. Express 1.0 rad, 0.6 rad, 7.25 rad, $\pi/2$ rad, and π rad in degrees.

Solution $45° = \frac{45}{360}(2\pi) = 0.78\,\text{rad}$ $\qquad 60° = \frac{60}{360}(2\pi) = 1.05\,\text{rad}$

$$90° = \frac{90}{360}(2\pi) = \frac{\pi}{2}\,\text{rad} = 1.57\,\text{rad} \qquad 170° = \frac{170}{360}(2\pi) = 2.97\,\text{rad}$$

$$1.0\,\text{rad} = \frac{1}{2\pi}(360°) = 57.3° \qquad 0.6\,\text{rad} = \frac{0.6}{2\pi}(360°) = 34.4°$$

$$7.25\,\text{rad} = \tfrac{7.25}{2\pi}(360°) = 415° = 55°\ (\text{subtract}\,360°)$$

$$\tfrac{\pi}{2}\ \text{rad} = \tfrac{0.5\pi}{2\pi}(360°) = 90° \qquad \pi\,\text{rad} = \tfrac{\pi}{2\pi}(360°) = 180°$$

The linear velocity in meters per second of a point as it moves around a circle is called the **tangential velocity:**

$$v = \tfrac{ds}{dt} = r\tfrac{d\theta}{dt}$$

We define the **angular velocity** ω in radians per second as $\omega = d\theta/dt$. Thus

$$\boxed{v = r\omega} \tag{10.2}$$

If the point is accelerating along its path with tangential acceleration a, then

$$a = \tfrac{dv}{dt} = r\tfrac{d\omega}{dt}$$

We define the **angular acceleration** α in radians per second, as $\alpha = d\omega/dt = d^2\theta/dt^2$. Thus

$$\boxed{a = r\alpha} \tag{10.3}$$

We have seen previously how to describe linear motion with constant acceleration. If we simply divide the earlier equations by r, we obtain the equations that describe the rotational motion. For example, the equation $v = v_0 + at$ becomes

$$\tfrac{v}{r} = \tfrac{v_0}{r} + \tfrac{at}{r} \qquad \text{or} \qquad \omega = \omega_0 + \alpha t$$

Table 10.1 summarizes the parallels between linear motion and rotation. Previously I used the variable x to represent displacement along a straight line. Now I am using the letter s to remind you that the "linear" motion is along a curved arc. However, the motion is still one-dimensional.

Table 10.1

Rotational Motion (α = constant)	**Linear Motion (a = constant)**
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \frac{1}{2}(\omega_0 + \omega)t$	$s = \frac{1}{2}(v_0 + v)t$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$s = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v_2 = v_0^2 + 2as$

Angular velocity ω and angular acceleration α are actually vector quantities, but as long as we keep the axis of rotation fixed, we do not need to worry about their vector nature. To keep things simple, I will just consider a counterclockwise rotation (as viewed from above) as positive .

It is common to describe rotating objects by specifying their **frequency** of revolution in revolutions per second. Since 1 rev is 2π rad, then

$$\boxed{\omega = 2\pi f} \qquad (10.4)$$

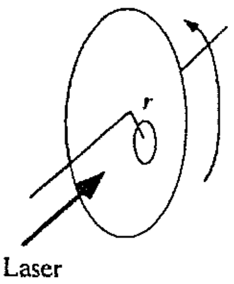
Equations 10.1 through 10.4 are important. **MEMORIZE** them. Sometimes one encounters rotation rates given in revolutions per minute. Be certain always to change to revolutions per second. Also, in using the above equations, be certain to use radians, not degrees. Be very careful about this or you will make errors.

Problem 10.1 An electric drill rotates at 1800 rev/min. Through what angle does it turn in 2 ms? If it reaches this speed from rest in 0.64 s, what is its average angular acceleration?

Solution $\theta = \omega t = 2\pi f t = (2\pi)(\frac{1800}{60\text{ s}})\,(0.002\text{ s}) = 0.37\text{ rad} = 22°$

$$\alpha = \tfrac{\omega}{t} = \tfrac{2\pi f}{t} = (2\pi)(\tfrac{1800}{60\text{s}})(\tfrac{1}{0.64\text{s}}) = 295\,\text{rad/s}^2$$

Problem 10.2 A light chopper consists of a disk spinning at 40 rev/s in which is cut a hole of diameter 1.0 mm, a distance of 5.4 cm from the axis. A very thin laser beam is directed through the hole parallel to the axis of the disk. The light travels at 3×10^8 m/s. What length of light beam is produced by the chopper?



Solution The hole is in front of the beam for a time t, where

$$d = vt = r\omega t, \qquad \text{so } t = \tfrac{d}{r\omega}$$

In time t the laser beam travels a distance of L:

$$L = ct = \tfrac{cd}{r\omega} = \tfrac{cd}{2\pi f r} = \tfrac{(3\times10^8\text{ m/s})(0.001\text{ m})}{(2\pi)(40\text{ rev/s})(0.054\text{ m})} = 2.2 \times 10^4\text{ m}$$

It is important to recognize that when a rigid body rotates, every point has the same angular velocity and the same angular acceleration. However, the linear speed and the tangential acceleration are not the same for all points. They increase for points farther from the axis. Note also that in addition to a possible tangential acceleration, each point has a centripetal acceleration directed inward. Centripetal acceleration and tangential acceleration are perpendicular vectors, and consequently the magnitude of the total acceleration is $a = \sqrt{a_c^2 + a_t^2}$.

10.2 ROTATIONAL KINETIC ENERGY

Imagine a rotating object to consist of lots of little pieces of mass. The piece m_i is at a distance r_i from the axis, and all rotate with the same angular velocity. The total kinetic energy associated with their rotational motion is

$$\begin{aligned} K_R &= \sum K_i = \sum \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} \sum m_i r_i^2 \omega^2 \\ &= \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 \end{aligned}$$

The quantity in parentheses is called the **moment of inertia.** A more descriptive name might be the "rotational mass," since it plays the same role for rotation that mass does for translational motion. 1/2 $I\omega^2$ is analogous to 1/2 mv^2.

$$\boxed{I = \sum m_i r_i^2} \qquad (10.5)$$

In terms of the moment of inertia the rotational kinetic energy can be expressed as

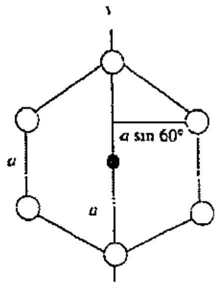
$$\boxed{K_R = \tfrac{1}{2} I \omega^2} \qquad (10.6)$$

A key feature of the moment of inertia is that it depends both on the amount of mass and on how far from the axis the mass is located. For example, when an ice skater spins with her arms in close to her body, she has a certain moment of inertia. If she then moves her arms outward, her moment of inertia increases appreciably, even though her mass does not change. This gives rise to some profound effects, as we shall see.

Problem 10.3 A molecule consists of identical atoms of mass m placed at each vertex of a regular hexagon of side a. Calculate the moment of inertia of the molecule about (a) the z axis that is perpendicular to the plane of the hexagon and passing through its center, and (b) the y axis that passes through two diametrically opposite atoms.

Solution All atoms are the same distance a from the z axis perpendicular to the plane, so

$$I_z = 6ma^2$$



To calculate I_y, note that two atoms are on the y axis, so $r = 0$ for them. For the other four atoms,

$$r = a\sin 60^\circ = \frac{\sqrt{3}}{2}a$$

Thus,

$$I_y = 4\,ma^2\left(\tfrac{3}{4}\right) = 3ma^2$$

Problem 10.4 As an alternative to the use of internal combustion engines, experimental cars have been designed that are propelled by the energy stored in a large spinning flywheel. Unfortunately, it turns out to be a challenging engineering problem to store enough energy in this way. The flywheels are huge and spinning so fast that they are hazardous. If you spin them too fast, the centripetal force required to hold the wheel together exceeds its breaking strength, and it can fracture and fly apart and go through your floorboard and blast you to kingdom come. A little golf-cart-type car might manage 20 mi/h using 5 hp. (a) Calculate the energy used to travel 20 mi. (b) Suppose you stored this much energy in a big steel flywheel with moment of inertia 18 kg $\cdot$ m^2. (Such a wheel might have a mass of 100 kg and a radius of 60 cm.) At what frequency, in revolutions per minute, would it have to rotate?

Solution To go 20 mi requires $1\,\text{h} = 3600\,\text{s}$.

(a) $$E = Pt = (5\,\text{hp})(746\,\text{W/hp})(3600\,\text{s}) = 1.3\times 10^7\,\text{J}$$

(b) $$\text{KE} = \tfrac{1}{2}I\omega^2 \qquad \omega = 2\pi f = (2\,\text{KE}/I)^{1/2} \qquad f = 194\,\text{s}^{-1} = 11{,}700\,\text{rev/min}$$

10.3 MOMENT OF INERTIA CALCULATIONS

Consider a continuous object to be composed of many small pieces of mass dm. Then Eq. 10.5 becomes

$$\boxed{I = \int r^2\,dm} \qquad (10.7)$$

If the mass is spread throughout the volume with density ρ (ρ = mass/volume), the mass in a volume dV is $dm = \rho\,dV$. For a surface mass density σ, $dm = \sigma\,dA$. For a line mass density λ, $dm = \lambda\,dx$. In these cases the moment of inertia can be written

$$\boxed{I = \int \rho r^2\,dV \quad \text{or} \quad I = \int \sigma r^2\,dA \quad \text{or} \quad I = \int \lambda x^2\,dx} \qquad (10.8)$$

Table 10.2 lists moments of inertias for some common shapes.

Problem 10.5 Calculate the moment of inertia of a hoop of mass M and radius R about its axis.

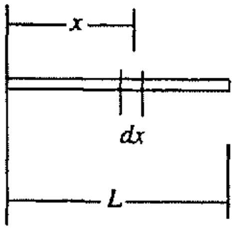
Solution All of the mass elements dm are the same distance R from the axis, so Eq. 10.7 yields

$$I = \int r^2\,dm = R^2 \int dm = MR^2$$

This result applies to a hollow cylinder (like a pipe) as well.

Problem 10.6 Calculate the moment of inertia of a uniform rod of mass M and length L rotated about an axis perpendicular to the rod and passing through one end.

The linear mass density is $\lambda = M/L$, so Eq. 10.8 yields

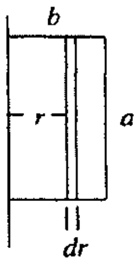


$$I = \int \lambda x^2\,dx = \int_0^L \frac{M}{L} x^2\,dx = \frac{1}{3}ML^2$$

Table 10.2. Moment of Intertia, I

Hoop or cylindrical shell about its axis	MR^2
Solid cylinder or disk	$\frac{1}{2}MR^2$
Rod about perpendicular axis through center	$\frac{1}{12}MR^2$
Rod about perpendicular axis through end	$\frac{1}{3}MR^2$
Rectangular plate $a \times b$ about perpendicular axis though center	$\frac{1}{12}M(a^2+b^2)$
Solid sphere	$\frac{2}{5}MR^2$
Spherical shell	$\frac{2}{3}MR^2$

Problem 10.7 Calculate the moment of inertia of a rectangular sheet of metal of mass M and of sides a and b about the edge of length a.



Solution

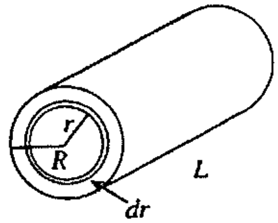
$$\sigma = \frac{M}{b} \quad I = \int_0^b \left(\frac{M}{ab}\right)(r^2)(a\,dr) = \frac{1}{3}Mb^2$$

Note that the result does not depend on the length a.

Problem 10.8 Calculate the moment of inertia of a solid cylinder of mass M and radius R about its axis.

Solution $\rho = \frac{M}{\pi R^2 L}$ $dV = 2\pi r dr L$

$$I = \int_0^R \left(\frac{M}{\pi R^2 L}\right)(r^2)(2\pi r L dr) = \tfrac{1}{2}MR^2$$



Parallel-Axis Theorem (prove this for yourself): If the moment of inertia about an axis passing through the center of mass is I_{CM}, then the moment of inertia about a parallel axis displaced by a distance d from the center of mass axis is

$$\boxed{I = I_{\mathrm{CM}} + Md^2} \tag{10.9}$$

Problem 10.9 The moment of inertia of a rod of mass M and length L about a perpendicular axis through its end is 1/3 MR^2. What is I about a parallel axis through the midpoint (the CM)?

Solution $I_{\mathrm{end}} = I_{\mathrm{CM}} + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2,$ so $I_{\mathrm{CM}} = \frac{1}{12}ML^2$

Problem 10.10 The moment of inertia of a hoop about an axis through the center and perpendicular to the plane of the hoop is MR^2. What is I for a parallel axis through a point on the hoop?

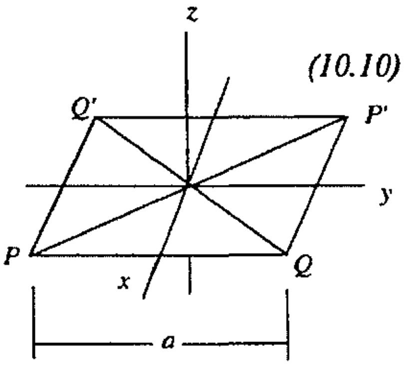
Solution Direct calculation is complicated, but using our theorem, it is easy to find I.

$$I = I_{\mathrm{CM}} + MR^2 = MR^2 + MR^2 = 2\,MR^2$$

Perpendicular-Axis Theorem (prove this for yourself): The moment of inertia of a plane object about an axis perpendicular to the plane is equal to the sum of the moments of inertia about any two perpendicular axes in the plane. Thus if the x and y axes are in the plane,

$$I_z = I_x + I_y \tag{10.10}$$

Problem 10.11 A square planar object has side a and mass M. What is its moment of inertia about (a) an axis through the center and perpendicular to the plane, and (b) an axis through diagonally opposite corners of the square? Use the previously obtained result (Problem 10.7) that the moment of inertia of a square about one edge is 1/3 Ma^2.



Solution Let the xy axes lie in the plane, passing through the center and parallel to the edges. From the parallel-axis theorem we find I_x.

$$I_{\text{edge}} = I_x + M\left(\tfrac{a}{2}\right)^2 \qquad \frac{Ma^2}{3} = I_x + \frac{Ma^2}{4} \quad \text{so } I_x = \frac{Ma^2}{12}$$

By symmetry, $I_x = I_y$ so

$$I_z = I_x + I_y = \frac{Ma^2}{12} + \frac{Ma^2}{12} = \frac{Ma^2}{6}$$

Axes PP' and QQ' are perpendicular and symmetric, so $I_P = I_Q$. From the perpendicular-axis theorem, $I_z = I_P + I_Q = 2I_P$. So

$$I_p = \tfrac{1}{2} I_z = \frac{Ma^2}{12}$$

This approach is much easier than calculating I_P directly.

10.4 TORQUE

When a net force is applied to an object, it acquires a linear acceleration. The rotational quantity analogous to force is **torque**. For an object to acquire an angular acceleration, it must be subject to a net torque. *Torque* means "twist." The torque due a force F about a pivot P is τ, where the magnitude of the torque is

$$\boxed{\tau = Fr \sin\,\theta = Fr_\perp} \qquad (10.11)$$

The distance from the pivot to the point of application of the force, as illustrated in Figure 10.2, is r. The term $r_\perp = r \sin\,\theta$ is the **perpendicular lever arm** (also called simply the "lever arm" or "moment arm"). It is the shortest distance from the pivot to the line of action of the force. Torque is a vector, and its more complete definition is

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}} \qquad (10.12)$$

τ is perpendicular to the plane of **F** and **r**. To find its direction, place the fingers of your right hand along **r**. Curl them toward **F**. Your thumb will point up along τ.

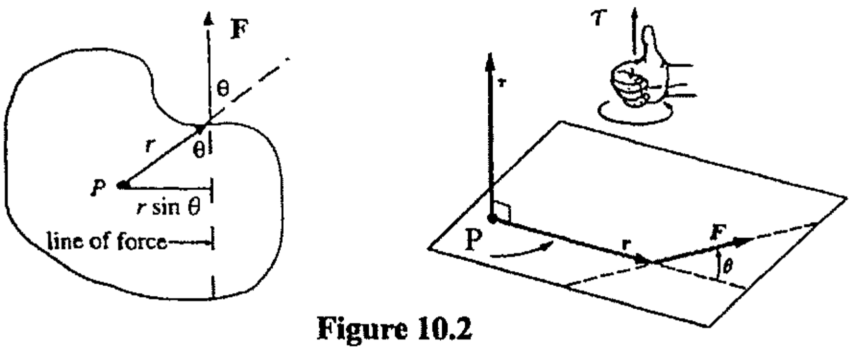


Figure 10.2

Now consider a particle of mass m moving in a circle of radius r. Suppose a tangential force F acts on the particle and accelerates it. Then

$$F = ma \qquad \text{or} \qquad Fr = mra$$

$Fr = \tau$, the torque acting on the particle, and $a = r\alpha$, where α is the angular acceleration of the particle. Thus for a single particle, $\tau = mr^2\alpha$.

Now consider an extended object (like a disk) that can rotate. Imagine it consists of many small elements dm. All of them rotate with the same angular acceleration, so if the external force acting on each element is dF, $dF = adm$, and $d\tau = rdF = radm = dmr^2\alpha$. Integrate to find the total external torque about the pivot P.

$$\tau_{\text{net}} = \int r^2\, dm\alpha = \alpha \int r^2\, dm = I\alpha$$

Thus

$$\boxed{\tau_{\text{net}} = I\alpha} \tag{10.13}$$

This equation is analogous to $F_{\text{net}} = ma$ for translational motion.

Problem 10.12 A string of negligible weight is wrapped around a pulley of mass M and radius R and tied to a mass m. The mass is released from rest, and it drops a distance h to the floor. Use energy principles to determine the speed of the mass when it hits the floor. Also, use Eq. 10.12 to determine this speed, as well as the tension in the string and the angular acceleration of the pulley.

Solution

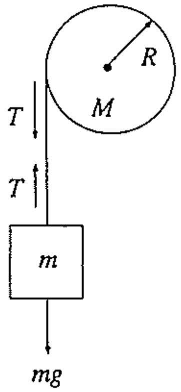
$$\text{KE}_1 + \text{PE}_1 = \text{KE}_2 + \text{PE}_2$$

$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 + 0$$

$$I = \tfrac{1}{2}MR^2 \qquad \text{and} \qquad v = R\omega$$

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)(\tfrac{v}{R})^2$$

$$v = (\frac{4gh}{2m + M})^{1/2}$$



Applying Eq. 10.12 to the pulley yields $\tau = I\alpha$ so

$$TR = \tfrac{1}{2}MR^2\alpha \qquad \alpha = \frac{2T}{MR}$$

Applying $F = ma$ to the mass m yields $T - mg = -ma$. Here a is the magnitude of the linear acceleration. I took up as positive (but acceleration is downward). But

$$a = \alpha R = (\tfrac{2T}{MR})R = \tfrac{2T}{M}, \text{ so } T - mg = -(m)(\tfrac{2T}{M})$$

$$T = (\frac{M}{M + 2m})mg \qquad a = \frac{2}{M + 2m}g \qquad \alpha = \frac{a}{R} = \frac{2g}{R(M + 2m)}$$

Since $v^2 = v_0^2 + 2ah = 0 + 2\left(\frac{2g}{M+2m}\right)h$, then $v = \left(\frac{4gh}{M+2m}\right)^{1/2}$

10.5 ROLLING

When a wheel of radius R rolls without slipping, a point on the circumference moves a distance ds when the wheel rotates through an angle $d\theta$, where $ds = Rd\theta$. If this happens in time dt,

$$\frac{ds}{d\theta} = R\frac{d\theta}{dt} \qquad \text{or} \qquad v = R\omega \tag{10.14}$$

One can imagine the motion to consist of simple rotation about the point of contact with the ground. Viewed in this way, the kinetic energy is rotational kinetic energy, where

$$\mathrm{KE} = \tfrac{1}{2}I\omega^2$$

By the parallel-axis theorem, Eq. 10.9, $I = I_{\mathrm{CM}} + MR^2$. Thus

$$\mathrm{KE} = \tfrac{1}{2}(I_{\mathrm{CM}} + MR^2)\omega^2 = \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2}MR^2\omega^2$$

Since $v = R\omega$, then

$$\boxed{\mathrm{KE} = \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2}Mv^2} \tag{10.15}$$

This is an important result. It states that the kinetic energy of a rolling object is equal to the kinetic energy of translation of the center of mass (imagining all of the mass concentrated there) plus the kinetic energy of rotation about the CM.

Problem 10.13 A hoop of radius R_H and mass m_H and a solid cylinder of radius R_C and mass m_C are released simultaneously at the top of a plane ramp of length L inclined at angle θ above horizontal. Which reaches the bottom first, and what is the speed of each there?

Solution The moment of inertia for each object is of the form $I_{\mathrm{CM}} = kmR^2$, where $k = 1$ for a hoop and $k = 1/2$ for a solid cylinder. Energy is conserved, so

$$mgh = \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2}mv^2 \qquad mgh = \tfrac{1}{2}kmR^2\omega^2 + \tfrac{1}{2}mv^2$$

$$v = R\omega, \text{ so } v = \sqrt{\frac{2gh}{1+k}} \text{ where } h = L\sin\theta$$

Thus

$$v_H = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh} \qquad \text{and} \qquad v_C = \sqrt{\frac{2gh}{1+1/2}} = \sqrt{\frac{4gh}{3}}$$

Thus we see that any solid cylinder will roll faster than a hoop of any size. Amazing.

10.6 ROTATIONAL WORK AND POWER

Suppose that a tangential force pushes on an object and causes it to rotate through an angle $d\theta$. If the distance from the pivot to the point of application of the force is r, the work done by the force is $dW = Fds = Frd\theta = \tau d\theta$. Thus

$$\boxed{W = \int \tau d\theta} \qquad (10.16)$$

If this work is done in time dt, the power is

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} \quad \text{or} \quad \boxed{P = \tau\omega} \qquad (10.17)$$

The work-energy theorem for rotational motion is

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau d\theta = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2} \qquad (10.18)$$

Problem 10.14 A molecule in a microwave oven experiences a torque $\tau = \tau_0 \sin\theta$. How much work must be done to rotate the molecule from $\theta = 0°$ to $\theta = 180°$?

Solution $W = \int_0^\pi \tau d\theta = \int_0^\pi \tau_0 \sin\theta d\theta = -\tau_0 \omega s\theta \int_0^\pi = -\tau_0[-1 - (-1)] = 2\tau_0$

10.7 SUMMARY OF KEY EQUATIONS

Instantaneous angular velocity: $\omega = \frac{d\theta}{dt}$ $\quad s = r\theta$

$v = r\omega$

Instantaneous angular acceleration: $\alpha = \frac{d\omega}{dt}$ $\quad a = r\alpha$

ANALOGOUS LINEAR AND ANGULAR QUANTITIES:

Linear impulse	$\bar{F}\Delta t$	↔	Angular impulse	$\bar{\tau}\Delta t$
Linear displacement	s	↔	Angular displacement	θ
Linear speed	v	↔	Angular speed	ω
Linear acceleration	a	↔	Angular acceleration	α
Mass (inertia)	m	↔	Moment of inertia	I
Force	F	↔	Torque	τ

If, in the equations for linear motion, we replace the linear quantities by the corresponding angular quantities, we get the corresponding equations for angular motion.

Linear:	$F = ma$	$\text{KE} = \frac{1}{2}mv^2$	$\text{Work} = Fs$	$\text{Power} = Fv$
Angular:	$\tau = I\alpha$	$\text{KE} = \frac{1}{2}I\omega^2$	$\text{Work} = \tau\theta$	$\text{Power} = \tau\omega$

In these equations, θ, ω, and α must be expressed in radians.

Motion with constant angular acceleration:

$$\theta = \theta_0 + \omega t + \tfrac{1}{2}\alpha t^2 \qquad \omega = \omega_0 + \alpha t \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Acceleration of particle on rotating body: $a_{\text{tan}} = R\alpha$ $\qquad a_{\text{cent}} = R\omega^2$

Moment of inertia: $I = \sum_{i=1}^{n} m_i R_i^2 \qquad I = \int \rho R^2 \, dV$

Parallel-axis theorem: $I = I_{\text{CM}} + Md^2$

Perpendicular-axis theorem (for a flat plate in the xy plane): $I_z = I_x + I_y$

Kinetic energy of rotation: $K_R = \tfrac{1}{2} I\omega^2$

Torque (direction is given by right-hand rule):

$$\tau = \mathbf{r} \times \mathbf{F} \qquad \tau = rF \sin\theta = r_\perp F$$

Rolling kinetic energy: $\text{KE} = \tfrac{1}{2} I_{\text{CM}}\omega^2 + \tfrac{1}{2} Mv^2$

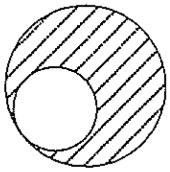
Supplementary Problems

SP 10.1 A disk initially at rest is given an angular acceleration of 12.0 rad/s^2. What is its angular velocity after 10 s? What is its frequency then in revolutions per minute (rev/min)? How many revolutions does it make during this time?

SP 10.2 A car accelerates from rest to 20 m/s in 8 s. The wheels have radius 0.32 m. What is the average angular acceleration of the wheels?

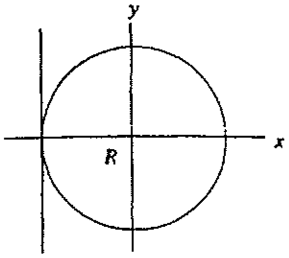
SP 10.3 A tether ball of mass 0.80 kg is attached to the top of a tall pole by a light cord 1.8 m long. What is the kinetic energy of the ball when the string makes an angle of 30° with vertical and the ball is rotating at 0.40 rev/s?

SP 10.4 A circular disk of radius R has mass M. A hole of diameter R is cut in the disk, positioned as shown here. What is the moment of inertia for rotations about an axis perpendicular to the disk and passing through its center?

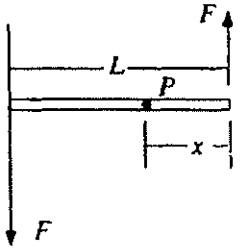


SP 10.5 A playground merry-go-round is a metal disk of radius 2.5 m and mass 80 kg. Two kids, each of mass 40 kg, are riding on the outer rim of the disk. When they move halfway in toward the center, by what factor does the moment of inertia change?

SP 10.6 What is the moment of inertia of a circular hoop of mass M and radius R about an axis that is tangent to the hoop and lies in its plane?



SP 10.7 Two oppositely directed forces of equal magnitudes are applied perpendicularly to the ends of a rod of length L. The rod is pivoted a distance x from one end. What is the torque about the pivot P? Does the result depend on the value of x?



SP 10.8 The head bolts on an engine must be tightened to the manufacturer's specification for proper operation. What is the minimum force you must apply to a 10-in wrench (25 cm) in order to exert a torque of 25 ft · lb (34 N · m)?

SP 10.9 A bicycle chain passes over a front sprocket with 32 teeth and a rear sprocket of 16 teeth. The crank arm on which the bicyclist pushes is 15 cm long, and the bike wheels have a radius of 33 cm. (a) When the cyclist pushes with 80 N on the pedal, what force is applied to the ground by the rear tire? (b) When the pedals make 4 rev/s, how fast does the bicycle move?

SP 10.10 A sphere of mass 0.036 kg and radius 1.2 cm rolls down an inclined plane. It is initially moving 0.48 m/s. How fast will it be moving after it has dropped 12 cm in elevation?

SP 10.11 A streetcar is to be powered by the energy stored in a flywheel of mass 240 kg and radius 0.80 m. The flywheel is initially rotating at 4000 rev/min. How long could the flywheel provide power at a rate of 10 hp?

Solutions to Supplementary Problems

SP 10.1
$$\omega = \omega_0 + \alpha t = 0 + (12\ \text{rad/s}^2)(10\ \text{s}) = 120\ \text{rad/s}$$

$$\omega = 2\pi f \qquad f = \frac{120\,\text{rad/s}}{2\pi} = 19.1\,\text{rev/s} = (19.1\,\text{rev/s})\ (60\ \text{s/min}) = 1150\,\text{rev/min}$$

$$\theta = \omega t = (120\ \text{rad/s})(10\ \text{s}) = 1200\ \text{rad} = 120\left(\tfrac{1}{2\pi}\tfrac{\text{rev}}{\text{rad}}\right) = 19.1\ \text{rev}$$

SP 10.2 $\omega = \frac{v}{r} = \dfrac{20\,\text{m/s}}{0.32\,\text{m}} = 62.5\,\text{rad/s} \qquad \omega = \omega_0 + \alpha t = 0 + \alpha t \qquad \alpha = \frac{\omega}{t} = \dfrac{62.5\,\text{rad/s}}{8\text{s}} = 7.8\ \text{rad/s}^2$

SP 10.3 The ball moves in a circle of radius $L \sin\ 30°$: $\omega = 2\pi f$, and $I = mr^2$.

$$\text{KE} = \tfrac{1}{2}I\omega^2 = (0.5)(0.80)(1.8\ \sin\ 30°)^2\,(2\pi)^2\,(0.40)^2\ \text{J}$$

$$= 2.05\ \text{J}$$

SP 10.4 Treat the hole as a small disk of negative mass superimposed on the larger solid disk (with no hole in it). Use the parallel-axis theorem to find I for the small disk. The mass of the small disk is

$$m = \frac{\pi(R/2)^2}{\pi R^2}M \qquad m = \tfrac{1}{4}M$$

$$I = I_{\text{big}} - I_{\text{small}} \quad \text{(since small disk has "negative" mass)}$$

$$= \tfrac{1}{2}MR^2 - \left[\tfrac{1}{2} + \left(\tfrac{M}{4}\right)\left(\tfrac{R}{2}\right)^2\right] = \tfrac{13}{32}MR^2$$

SP 10.5 The moment of inertia of the disk is $I_D = 1/2\ MR^2$. The moment of inertia of the two kids is $I_K = 2mr^2$. Thus

$$\frac{I_2}{I_1} = \frac{I_D + I_{K2}}{I_D + I_{K1}} = \frac{1/2\,MR^2 + 2m(R/2)^2}{1/2\,MR^2 + 2MR^2} = \frac{M+m}{M+4m} = \frac{1}{2}$$

SP 10.6 The moment of inertia about the axis of the hoop (through its center) is MR^2. By the perpendicular-axis theorem, $I_2 = I_x + I_y = MR^2$. By symmetry, $I_x = I_y$. Thus, $MR^2 = 2I_x$, $I_x = 1/2\ MR^2$. By the parallel-axis theorem, $I = I_x + MR^2 = 3/2\ MR^2$.

SP 10.7 $\tau = F(L-x) + Fx = FL$ independent of x. A pair of forces like this is called a "couple." The net torque exerted is independent of where the pivot is located.

SP 10.8 $$\tau = Fr_\perp \qquad F = \frac{\tau}{r_\perp} = \frac{34\,\text{N·m}}{0.25\,\text{m}} = 136\,\text{N} = 31\,\text{lb}$$

SP 10.9 (a) Suppose the pedal makes 1 rev. The work done is $W_1 = 2\pi rF_1$. The back sprocket then makes 0.5 rev, and the tire moves a distance of πR along the ground, exerting a force F_2. Thus

$$2\pi rF_1 = \pi RF_2 \qquad F_2 = \frac{r}{R}F_1 = \left(\tfrac{15}{33}\right)(80\text{N}) = 36\text{N}$$

(b) In 1 s the pedals make 4 rev, so the back wheel makes (16/32)(4) = 2 rev, and the bike travels a distance of $s = (2)(2\pi R_2) = (4\pi)(0.33\text{m}) = 2.1\,\text{m}$, so $v = 2.1$ m/s.

SP 10.10 $$\tfrac{1}{2}I\omega_1^2 + \tfrac{1}{2}mv_1^2 + mgh = \tfrac{1}{2}I\omega_2^2 + \tfrac{1}{2}mv_2^2 \qquad I = \tfrac{2}{5}mv^2 \qquad v = w$$

$$v_2 = \sqrt{v_1^2 + 10/7\ gh} = \sqrt{(0.48\text{m/s})^2 + (10/7)(9.8\text{m/s}^2)(0.12\,\text{m})} = 1.38\ \text{m/s}$$

Note that mass and radius do not affect the answer. All spheres roll at the same rate.

SP 10.11 $$P = \frac{E}{t} \qquad t = \frac{E}{P} \qquad E = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}MR^2\omega^2$$

$$t = \frac{MR^2\omega^2}{2P} = \frac{(240)\,(0.8)^2\,(2\pi)^2\,(4000/60)^2}{(2)\,(10)\,(746)}$$

$$= \frac{(240\,\text{kg})\,(0.8\,\text{m})^2\,(2\pi)^2\,(4000/60\text{s})^2}{(2)\,(10\,\text{hp})\,(746\,W/\text{h})} = 7.5\,\text{s}$$

Chapter 11

Angular Momentum

Many aspects of rotational motion are analogous to translational motion. However, some rotational phenomena are bizarre (for example, gyroscope motion) and seem almost magical when first encountered. The theory of angular momentum has profound consequences in quantum mechanics and all of modern physics, and it has led to our understanding of atoms.

11.1 ANGULAR MOMENTUM AND TORQUE

The angular momentum with respect to the origin of a particle with position **r** and momentum $\mathbf{p} = m\mathbf{v}$ is

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}} \qquad (11.1)$$

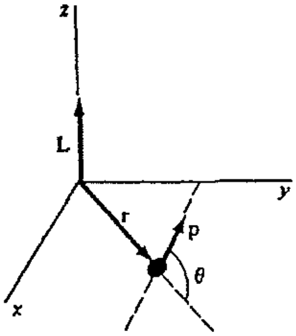


Figure 11.1

If the angle between **r** and **p** is θ, then the magnitude of L is

$$\boxed{L = rp \sin\theta = mvr \sin\theta} \qquad (11.2)$$

If **r** and **p** lie in the xy plane, **L** is along the z axis (Figure 11.1). The time rate of change of the angular momentum is

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad (11.3)$$

The cross product of a vector with itself is always zero, so

$$\frac{d\mathbf{r}}{dt} \times \mathbf{p} = \frac{d\mathbf{r}}{dt} \times m\mathbf{v} = m\,(\mathbf{v} \times \mathbf{v}) = 0, \quad \text{since } \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

From Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. Thus

$$\boxed{\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}} \quad \text{or} \quad \boxed{\frac{d\mathbf{L}}{dt} = \vec{\tau}} \qquad (11.4)$$

If we think of a rigid body as a collection of particles of mass m_i, the z component of the angular momentum can be expressed in terms of the moment of inertia, $I = \sum m_i r_i^2$.

$$L_z = \sum m_i r_i v_i = \sum m_i r_i^2 \omega \qquad \text{or} \qquad \boxed{L_z = I\omega} \tag{11.5}$$

This expression is analogous to $p = mv$, where I is like m and ω is like v. Apply Eq. 11.4 to a rigid body.

$$\frac{d\mathbf{L}}{dt} = \sum \mathbf{r}_i \times \mathbf{F}_i$$

All terms involving internal forces cancel, so

$$\frac{d\mathbf{L}}{dt} = \sum \mathbf{r}_i \times \mathbf{F}_{i,\text{ext}} \qquad \text{or} \qquad \boxed{\frac{d\mathbf{L}}{dt} = \vec{\tau}_{\text{ext}}} \tag{11.6}$$

Equation 11.6 is analogous to $\mathbf{F}_{\text{ext}} = d\mathbf{p}/dt$ for translational motion.

If no external torque acts on a system, the angular momentum of the system remains constant.

$$\boxed{\text{If } \tau = 0, \text{ then } \mathbf{L} = \text{constant.}} \tag{11.7}$$

This is the **law of conservation of angular momentum.**

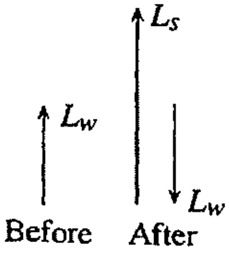
Problem 11.1 In an interesting lecture demonstration, a student sits in a swivel chair. She has moment of inertia I_s about a vertical axis. She holds vertical the axis of a bicycle wheel of moment of inertia $I_B \ll I_s$ spinning with large angular velocity ω_1. The wheel is spinning counterclockwise, as viewed from above. Now she rotates the wheel axis by 180°. What happens?

Solution No external torque acts (friction is negligible) so the angular momentum of the system remains constant. Initially $L_1 = I_w\omega_1$, directed upward. After the wheel axis is inverted, the angular momentum vector of the wheel will point downward, so the chair will rotate counterclockwise with angular velocity ω_2 so that the total angular momentum remains unchanged in magnitude and points upward.

$$L_1 = I_w\omega_1 \qquad L_2 = I_w(-\omega_1 + \omega_2) + I_s\omega_2$$

$$L_1 = L_2, \quad \text{so } I_w\omega_1 = I_w(-\omega_1 + \omega_2) + I_s\omega_2$$

$$\omega_2 = \frac{2I_w}{I_w + I_s}\omega_1$$



Problem 11.2 A disk is mounted with its axis vertical. It has radius R and mass M. It is initially at rest. A bullet of mass m and velocity v is fired horizontally and tangential

to the disk. It lodges in the perimeter of the disk. What angular velocity will the disk acquire?

Solution Angular momentum is conserved. The initial angular momentum is just the angular momentum of the bullet, $L_1 = mvR$. After the collision

$$L_2 = mR^2\omega + I\omega \qquad \text{where } I = \tfrac{1}{2}MR^2 \text{ and } L_1 = L_2, \qquad \text{so } \omega = \frac{mv}{(M+2m)R}$$

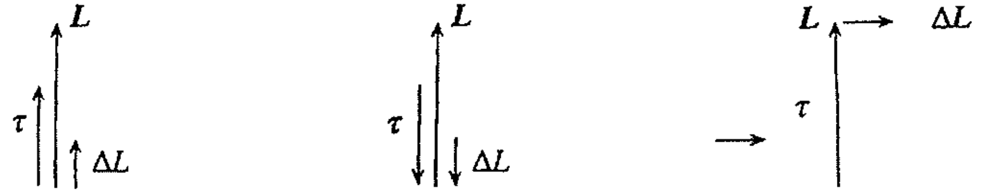
Problem 11.3 The force of gravity on the earth due to the sun exerts negligible torque on the earth (assuming the earth to be spherical), since this force is directed along the line joining the centers of the two bodies. The earth travels in a slightly elliptical orbit around the sun. When it is nearest the sun (the perihelion position), it is 1.47×10^8 km from the sun and traveling 30.3 km/s. The earth's farthest distance from the sun (aphelion) is 1.52×10^8 km. How fast is the earth moving at aphelion?

Solution No torque acts, so the earth's angular momentum is constant.

$$mr_1v_1 = mr_2v_2 \qquad \text{or} \qquad v_2 = \frac{r_1}{r_2}v_1 = \frac{1.47 \times 10^8}{1.52 \times 10^8}(30.3\,\text{km/s}) = 29.3\,\text{km/s}$$

11.2 PRECESSION

When a torque acts on a system, the angular momentum **L** will change by an amount $\Delta\mathbf{L}$ in time Δt. From Eq. 11.4, $\mathbf{L} = \vec{\tau}\Delta t$, so the change in angular momentum is a vector directed in the direction of the torque. For a rigid body, where $L = I\omega$ and I is fixed, an increase in angular momentum (longer **L** vector) means the body speeds up and rotates more rapidly. A smaller **L** means the body slowed down. A change in direction of **L** means that the axis of rotation has changed direction. Three simple cases are shown in Figure 11.2.



τ Parallel to **L**
Rotates Faster

τ Antiparallel to **L**
Rotates Slower

τ Perpendicular to **L**
Precesses

Figure 11.2

When the torque is directed perpendicular to **L**, Δ**L** is also perpendicular to **L**. This means that **L** will rotate without changing magnitude. A spinning top exhibits this behavior. Gravity acts on the center of mass of the top and exerts a downward force (the weight) which produces the torque τ shown in Figure 11.3. Think of the angular momentum vector **L** as "chasing" the torque vector. In Figure 11.4 you can see that **L** rotates about the z axis through a small angle $\Delta\phi$ in time Δt. This rotation about the z axis is called **precession**. From Figure 11.4 we can deduce an expression for the precessional frequency.

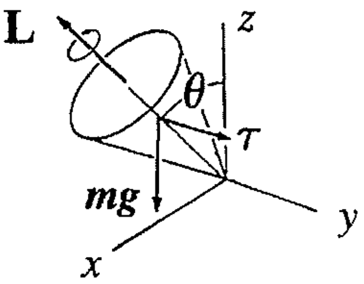


Figure 11.3

For small $\Delta\phi$,

$$\Delta L = L \sin\theta \, \Delta\phi$$

Divide by Δt:

$$\frac{\Delta \mathbf{L}}{\Delta t} = L \sin\,\theta \frac{\Delta\phi}{\Delta t}$$

$$= \vec{\tau}$$

and $\omega_p = \Delta\phi / \Delta t$ = angular frequency of precession.

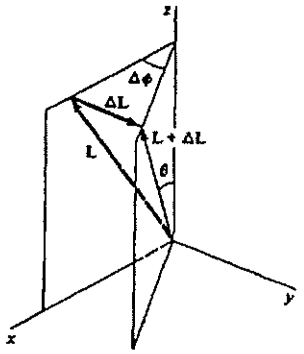


Figure 11.4

Thus $$\omega_p = \frac{\tau}{L\,sin\,\theta} \qquad (11.8)$$

In the above I assumed the **spin angular momentum**, $I\omega$, is large compared to the angular momentum associated with the precession. The faster the top spins, the more slowly it precesses. Even though the top is leaning to one side and would quickly fall were it not spinning, its spinning causes its axis to sweep out the surface of a cone as it precesses around the z axis (Figure 11.15). Amazing! In real tops the spin angular momentum is usually not large enough to justify the above approximation, and the motion consists of precession plus some complicated wobbling (called *nutation*). Thick books are written on the complex and fascinating motion of tops and gyroscopes.

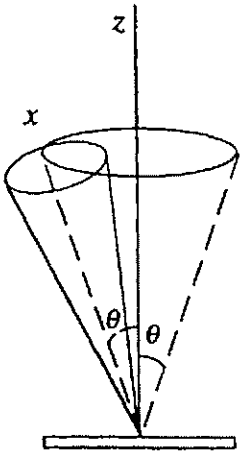
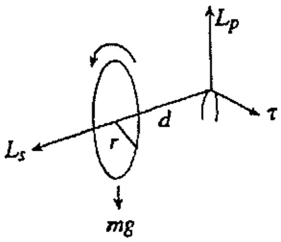


Figure 11.5

Problem 11.4 A weighted bicycle wheel is adapted to have a long axle. The axle is approximately horizontal and is supported at a point $d = 18\,\text{cm}$ from the wheel. Essentially all of the mass of the wheel (13.2 kg) is at the rim a radius $r = 32.0\,\text{cm}$ from the axle. The wheel is spinning at 240 rev/min. What is the angular velocity of precession? Compare the spin angular momentum with the angular momentum associated with the precession.



Solution $L \simeq L_s = I\omega \qquad \tau = mgd$

$I = mr^2$ for a hoop. The precession frequency is given by Eq. 11.8, with $\theta = 90°$.

$$\omega_p = \frac{mgd}{mr^2\omega} = \frac{gd}{r^2\omega} = \frac{(9.8)(0.18)}{(0.32)^2(2\pi)^2(240/60)^2} = 0.027\,\text{rad/s} = 0.26\,\text{rev/min}$$

The angular momentum due to precession is

$$L_p = md^2\omega_p = \frac{mgd^3}{r^2\omega}$$

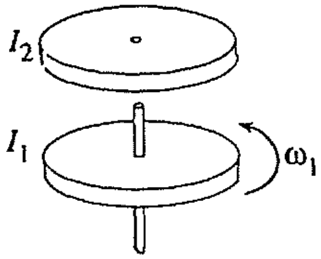
$$\frac{L_p}{L_s} = \frac{mgd^3}{mr^4\omega^2} = \frac{gd^3}{r^4\omega^2} = \frac{(9.8)(0.18)^3}{(0.32)^4(2\pi)^2(240/60)^2} = 8.6 \times 10^{-3}$$

Thus $$L_p \ll L_s$$

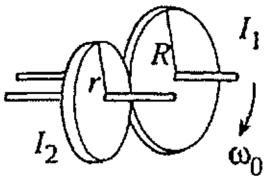
Supplementary Problems

SP 11.1 At an instant when a particle of mass m is at the position (2, 3, –4), its velocity has components (1, –1, 3). What is the angular momentum of the particle with respect to the origin?

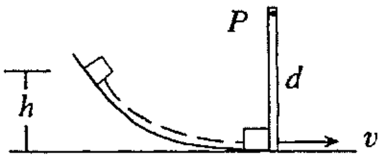
SP 11.2 An arrangement encountered in disk brakes and certain types of clutches is shown here. The lower disk, of moment of inertia I_1, is rotating with angular velocity ω_1. The upper disk, with moment of inertia I_2, is lowered onto the bottom disk. Friction causes the two disks to adhere, and they finally rotate with the same angular velocity. Determine this final angular velocity if the initial angular velocity of the upper disk was (a) zero, (b) ω_2 in the same direction as ω_1, and (c) ω_2 in the opposite direction from ω_1.



SP 11.3 A disk of radius R and moment of inertia I_1 rotates with angular velocity ω_0. The axis of a second disk, of radius r and moment of inertia I_2, is at rest. The axes of the two disks are parallel. The disks are moved together so that they touch. After some initial slipping the two disks rotate together. Find the final rate of rotation of the smaller disk.



SP 11.4 A small mass m slides from rest down a smooth slope, dropping a distance h in elevation. It strikes the end of a rod of length d and mass M and sticks to it. The rod is

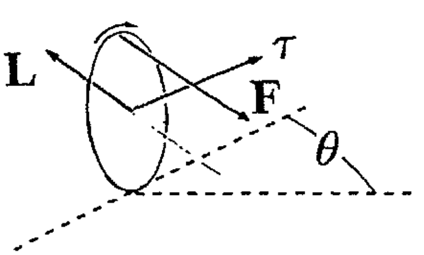


initially at rest and fastened by a pivot P. How fast is the small mass moving just after the collision?

SP 11.5 Show that the precessional frequency of a top (given in Eq. 11.8), is independent of θ, the angle between vertical and the spin axis of the top.

SP 11.6 If the pilot of a small single-engine airplane is gliding with the engine off and pulls back on the control stick, the flaps will create a torque that makes the front of the plane rise. Suppose, however, that the engine is running and the propeller is turning clockwise, as viewed by the pilot. The propeller and engine now have significant angular momentum. Explain what will happen now if the pilot pulls back on the stick. What happens if he pushes forward on the stick, in an attempt to lose elevation?

SP 11.7 If you played with Hula Hoops when you were a kid, you probably learned this trick. If you want to make the hoop turn to the right, run along behind it with a small stick in your hand. Give the top of the hoop a sharp blow directed to the right and parallel to the ground. This impulse will cause the axis of the hoop to rotate about a vertical axis, and the hoop will roll on at an angle θ to its original direction. Determine θ as a function of the average force F you apply and the duration Δt of the impulse. The hoop has mass M, radius R, and speed v. It does not slip on the ground.



SP 11.8 The evolution of a star depends on its size. If a star is sufficiently large, the gravity forces holding it together may be large enough to collapse it into a very dense object composed mostly of neutrons. The density of such a neutron star is about 10^{14} times that of the earth. Suppose that a star initially had a radius about that of our sun, 7×10^5 km, and that it rotated once every 26 days, as our sun does. What would be the period of rotation (the time for 1 rev) if the star collapsed to a radius of 15 km?

SP 11.9 A comet of mass 2×10^{14} kg moves in an elliptical orbit about the sun. The sun is at one focus of the elliptical orbit. At its point of closest approach to the sun, approximately 10^6 km, the comet is moving 5×10^6 m/s. What is the angular momentum of the planet with respect to an origin at the sun?

SP 11.10 A hockey stick of mass m and length d lies on the ice. It is struck perpendicularly a distance of $d/5$ from one end by a hockey puck. In the collision an impulse $F\Delta t$ is applied to the stick. Describe the subsequent motion of the stick. Assume the stick is a uniform rod.

SP 11.11 A bowling ball of mass m and radius r is launched with speed v_0 (and with no spin) on an alley where the coefficient of friction is μ. How far will it travel before it rolls without slipping? What then will be its speed?

SP 11.12 When I was in graduate school in Berkeley, we were as poor as church mice. My wife used to go to the UC Dental School in San Francisco (it was cheap), and while she was there, I would take our boys to the merry-go-round across the street in Golden Gate Park. It consisted of a metal disk (90-kg mass, 2.4-m radius) mounted on a vertical axis. A great game is played like this: Position eight kids (20 kg each) on the outer edge of the merry-go-round. Get it going as fast as you can (I could manage about 1 rev every 2 s). Then scream "Banzai!" or "Geronimo" (anything conveying bravery works) at which signal the kids all scramble toward the center. The merry-go-round takes off like a flying saucer. Any kids remaining on the perimeter are hurled off and go tumbling head over heels. Mothers drop like flies with heart attacks at the sight. Suppose for the parameters above the kids all move in to a position 0.6 m from the axis. What then will be the frequency of revolution? Is the kinetic energy of the system conserved? Explain.

Solutions to Supplementary Problems

SP 11.1

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = m(2\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 3\,\mathbf{k})$$

$$= m\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 1 & -1 & +3 \end{vmatrix} = m(9-4)\mathbf{i} + m(-4-6)\,\mathbf{j} + m(-2-3)\mathbf{k}$$

$$= 5m\,\mathbf{i} - 10m\,\mathbf{j} - 6m\,\mathbf{k}$$

SP 11.2 (a)

$$L_1 = L_2 \qquad I_1\omega_1 = (I_1 + I_2)\omega \qquad \omega = \frac{I_1}{I_1 + I_2}\omega_1$$

(b)

$$L_1 = I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega \qquad \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(c)

$$L_1 = I_1\omega_1 - I_2\omega_2 = (I_1 + I_2)\omega \qquad \omega = \frac{I_1\omega_1 - I_2\omega_2}{I_1 + I_2}$$

SP 11.3 The contact force F has the same magnitude for each disk. Thus the torque equation becomes

$$Fr = I_2\alpha_2 \quad \text{and} \quad \omega_2 = \alpha_2 t = \frac{Frt}{I_2} \quad \text{so } Ft = \frac{I_2\omega_2}{r} \qquad (i)$$

$$FR = I_1\alpha_1 \qquad \omega_1 = \omega_0 - \alpha_1 t = \omega_0 - \frac{FRt}{I_1} \qquad (ii)$$

Since v at contact point is finally the same for both disks, $r\omega_2 = R\omega_1$. Substitute Eq. i in Eq. ii and solve.

$$\omega_2 = \frac{rRI_1}{r^2 I_1 + R^2 I_2}\omega_0$$

SP 11.4 Energy conserved sliding down: $mgh = 1/2\ mv^2$, and $v = \sqrt{2gh}$. Angular momentum conserved in collision: $mvd = mVd + I\omega$, where $V = \omega d$ and $I = 1/3\ Md^2$ (rod).

$$m\sqrt{2gh}\ d = mVd + \tfrac{1}{3}Md^2\omega \qquad V = \frac{3m\sqrt{2gh}}{3m + M}$$

SP 11.5 From Figure 11.3, $\tau = Wd\sin\theta$. From Eq. 11.8,

$$\omega_p = \frac{\tau}{L\sin\theta} = \frac{Wd\sin\theta}{L\sin\theta} = \frac{Wd}{L}$$

ω_p is thus independent of θ.

SP 11.6 If you pull back on the stick, this creates a torque vector directed to your right. The angular momentum vector of the propeller is directed straight ahead. It tends to move toward the torque vector, so when you pull back on the stick, the plane goes up and to the right. When you push forward on the stick, the plane goes down and to the left. This is a noticeable effect in a small plane.

SP 11.7 The force creates a torque vector directed straight ahead and parallel to the ground. The angular momentum vector of the rolling hoop is to the left. The hoop precesses about a vertical axis, with **L**

moving toward the torque vector. In time Δt it precesses through an angle $\Delta\theta$. The frequency of precession is given by Eq. 11.8, with $\theta = 90°$.

Note that if the hoop doesn't slip, the friction force of the ground exerts a reaction force equal to the force applied at the top of the hoop. This friction force gives a torque about the center of mass in the same direction as the torque due to the applied force. Thus

$$\tau = FR + FR = 2FR$$

and $$\Delta\theta = \omega_p \Delta t = \frac{\tau \Delta t}{L \sin\theta} = \frac{2FR\Delta t}{I\omega}, \qquad I = MR^2, \qquad \omega = \frac{v}{R}, \text{ so } \Delta\theta = \frac{2F\Delta t}{Mv}$$

SP 11.8 $$L_1 = L_2, \text{ so } I_1\omega_1 = I_2\omega_2$$

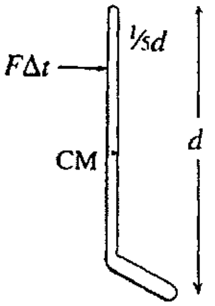
$$I = \frac{2}{5}MR^2, \qquad \omega = \frac{2\pi}{T}, \qquad \text{so } \frac{T_2}{T_1} = \frac{\omega_1}{\omega_2} = \frac{R_2^2}{R_1^2}$$

$$T_2 = \frac{R_2^2}{R_1^2}T_1 = \left(\frac{15}{7 \times 10^8}\right)^2 (26\,\text{days}) = 1.19 \times 10^{-14}\,\text{days} = 1.4\text{ms}$$

SP 11.9 $$L = mvd = (2 \times 10^{14}\,\text{kg})(5 \times 10^6\,\text{m/s})(10^9\,\text{m})$$

$$= 10^{30}\,\text{kg} \cdot \text{m/s}$$

SP 11.10 The center of mass of the stick acquires a velocity v, where



$$\Delta p = mv - 0 = F\,\Delta t \quad \text{or} \quad v = \frac{F\,\Delta t}{m}$$

The distance from the center of mass to the point of application of the force is

$$l = \frac{d}{2} - \frac{d}{5} = \frac{3d}{10}$$

Thus the torque applied about the CM is $\tau = 3/10\ Fd$. The change in angular momentum is thus

$$\Delta L = \tau \Delta t$$

$$= \frac{3}{10}d\ F\Delta t$$

$$= I\Delta\omega = I(\omega - 0) = I\omega$$

From Table 9.1, $I = 1/12\ md^2$ for a rod rotating about its center. Thus

$$\Delta L = \frac{1}{12}md^2\omega = \frac{3}{10}\ d\ F\ \Delta t \qquad \omega = \frac{18}{5}\frac{F\,\Delta t}{d}$$

Thus the CM of the stick slides with constant velocity v, and the stick rotates with constant angular velocity ω.

SP 11.11 The friction force will exert a force that causes the velocity to *decrease* at a constant rate, that is,

$$v = v_0 - at = v_0 - \frac{F_f}{m}t, \qquad F_f = \mu mg \quad \text{or} \quad v = v_0 - \mu g t$$

The friction force exerts a torque about the center of mass, $\tau = rF_f = r\mu mg$. This torque causes the angular momentum L (and also the angular velocity ω, where $L = I\omega$) to *increase* linearly. Thus

$$\omega = \omega_0 + \alpha t = 0 + \tfrac{\tau}{I}t = \frac{\mu mgr}{I}t.$$

For a sphere, $I = 2/5\ mr^2$, so

$$\omega = \tfrac{5}{2}\tfrac{\mu g}{r}t$$

When $v = r\omega$, the motion of the sphere will be pure rolling (until then it is also slipping). This will happen after time t, where

$$v_0 - \mu gt = r\left(\tfrac{5}{2}\tfrac{\mu g}{r}t\right) \qquad \text{or} \qquad t = \tfrac{3}{7}\tfrac{v_0}{\mu g}$$

The distance traveled in this time is

$$x = v_0t - \tfrac{1}{2}\mu gt^2$$

$$= v_0\left(\tfrac{2}{7}\tfrac{v_0}{\mu g}\right) - \tfrac{1}{2}\mu g\left(\tfrac{2}{7}\tfrac{v_0}{\mu g}\right)^2 = \tfrac{12}{49}\tfrac{v_0^2}{\mu g}$$

At this time the speed will be

$$v = v_0 - \mu gt$$

$$= v_0 - \mu g\left(\tfrac{2}{7}\tfrac{v_0}{\mu g}\right) = \tfrac{5}{7}v_0$$

Once the ball's motion is pure rolling, its speed doesn't change. Note also that the final speed is independent of the value of the coefficient of friction, although the time required to reach this final speed does depend on μ.

SP 11.12 No external torque acts, so the angular momentum of the system is conserved.

$$L_{\text{before}} = L_{\text{after}} \quad \text{or} \quad I_1\omega_1 = I_2\omega_2 \quad \omega = 2\pi f, \quad \text{so } f_2 = \frac{I_1}{I_2}f_1$$

Initially, $I_1 = 8mr^2 + 1/2\ Mr^2$ ($I = 1/2\ Mr^2$ for a disk). After moving in,

$$I_2 = 8m\left(\tfrac{r}{4}\right)^2 + \tfrac{1}{2}Mr^2$$

Thus

$$f_2 = \frac{8mr^2 + 0.5Mr^2}{8/16\ mr^2 + 1/2\ Mr^2}f_1 = \frac{16m + M}{m + M}f_1 \qquad f_2 = \frac{(16)(20) + 90}{20 + 90}(0.5\,\text{s}^{-1}) = 1.9\,\text{s}^{-1}$$

The merry-go-round speeds up by a factor of almost 4!

$$\text{KE}_1 = 8\left(\tfrac{1}{2}mr^2\omega_1^2\right) + \tfrac{1}{2}I\omega_1^2 \quad \left(I = \tfrac{1}{2}Mr^2\right) \quad \text{KE}_2 = 8\left[\tfrac{1}{2}m\left(\tfrac{r}{4}\right)^2\omega_2^2\right] + \tfrac{1}{2}I\omega_2^2$$

$$\frac{\text{KE}_2}{\text{KE}_1} = \frac{1/4\ mr^2\omega_2^2 + 1/4\ Mr^2\omega_2^2}{4mr^2\omega_1^2 + 1/4\ Mr^2\omega_1^2} = \frac{m + M}{16m + M}\left(\frac{f_2}{f_1}\right)^2 = \frac{m + M}{16m + M}\left(\frac{16m + M}{m + M}\right)^2 = \frac{16m + M}{m + M} = 3.7$$

Thus kinetic energy is *not* conserved. The kids must pull hard and do work to move in to the center (try it yourself), and this work increases the kinetic energy of the system.

Chapter 12

Statics and Elasticity

We have seen that when an object is treated as a point mass, the condition that it be in equilibrium (that is, remain at rest or move in a straight line with constant speed) is that the net external force acting on it is zero. However, real objects are extended bodies that can rotate, and so we now continue our idea of equilibrium to encompass this possibility.

12.1 ROTATIONAL EQUILIBRIUM

If the angular velocity of a rigid body is not to change, no net external torque can act on the object, since

$$\sum \vec{\tau}_i = \frac{d\mathbf{L}}{dt} = I\frac{d\omega}{dt}$$

Consider an object that is at rest, that is, not translating and not rotating. Since calculation of torque requires identification of an origin about which to calculate the torque, we might imagine that the requirement of zero net torque would be influenced by the choice of an origin. This is not so. If an object is not rotating, it is not rotating about *every* point, even points outside the object. This will become more clear in the examples that follow. A judicious choice of the origin will simplify the algebra, but any choice of origin will work. (I call the origin about which the torque is calculated the "imaginary pivot" to help you visualize what is happening.) Note that a force that passes through a point exerts no torque about that point since its lever arm is then zero. Thus it is helpful to choose a pivot through which unknown forces pass, since then they drop out of the calculation and reduce the number of unknown variables. The complete conditions for the equilibrium of a rigid body are as follows:

The resultant external force is zero: $$\boxed{\sum \mathbf{F} = 0} \qquad (12.1)$$

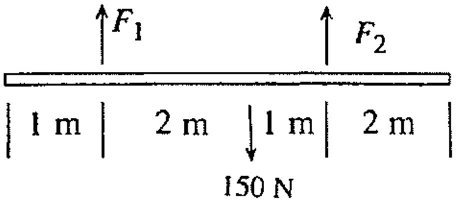
The resultant external torque is zero about any origin: $$\boxed{\sum \vec{\tau} = 0} \qquad (12.2)$$

In most of the following I will discuss the case of **statics**, where the object is at rest and not translating or rotating. In such problems we usually want to find the forces acting on various parts of the system or structure. It is essential that a careful and accurate force diagram be drawn, showing the external forces acting on the system. All or part of a structure can be considered the "system." Recall that torque = (force)(perpendicular lever arm). Thus $\tau = Fr\sin\theta = Fr_{\perp}$. Most often errors occur in drawing the force diagram and in determining the lever arms. Study the following examples carefully and you will avoid these pitfalls.

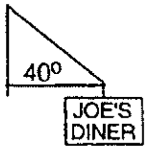
When dealing with objects whose mass is distributed, treat all of the mass as if it is concentrated at the center of mass (CM). The gravity force exerts no torque about the center of mass point, which is why I think of the CM as the "balance point."

Problem 12.1 Abe and Mary carry a uniform log of length 6 m and weight 150 N. Abe is 1 m from one end and Mary is 2 m from the other end. What weight does each person support?

Solution Draw the force diagram. If we imagine a pivot at Abe's position, and take a counterclockwise torque as positive, then Mary exerts a positive torque and the weight of the log exerts a negative torque. Using Eq. 12.2, $F_2(3\ \text{m}) - (150\,\text{N})(1\ \text{m}) = 0$, and $F_2 = 100\,\text{N}$. From Eq. 12.1, $F + F_2 = 150\,\text{N}$, so $F_1 = 50\,\text{N}$. If instead we had chosen the pivot at Mary's position, the torque equation would yield $-F_1(3\,\text{m}) + (150\,\text{N})(1\,\text{m}) = 0$, and $F_1 = 50\,\text{N}$ as above.

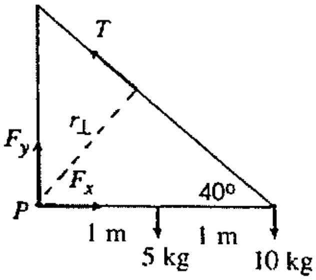


Problem 12.2 A 5-kg beam 2 m long is used to support a 10-kg sign by means of a cable attached to a building. What is the tension in the cable and compressive force exerted by the beam?

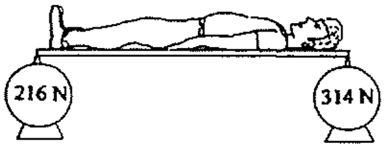


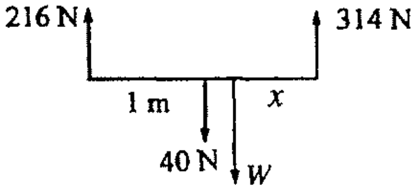
Solution Calculate the torque about a pivot at point P.

$$\begin{aligned} &r_{\perp} = 2\sin 40^\circ = 1.3\,\text{m} \\ &T\,(1.3\,\text{m}) - 5\,\text{kg}\,(1\,\text{m}) - 10\,\text{kg}\,(2\,\text{m}) = 0 \\ &T = 19\,\text{kg} = (19)(9.8) = 188\,\text{N} \\ &\textstyle\sum F_x = 0 \qquad F_x - T\cos 40^\circ = 0 \qquad F_x = 144\,\text{N} \end{aligned}$$



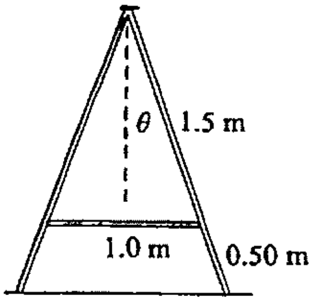
Problem 12.3 In kinesiology (the study of human motion), it is often useful to know the location of the center of mass of a person. This can be determined with the arrangement shown here. A plank of weight 40 N is placed on two scales separated by 2.0 m. A person lies on the plank and the left scale reads 314 N and the right scale reads 216 N. What is the distance from the left scale to the person's center of mass?



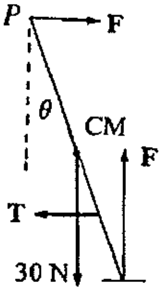


Solution Calculate the torque about the CM of the person: (216 N)$(2 - x)$ – (40 N)$(1 - x)$ – (314 N)$(x) = 0$ and $x = 0.80\,\text{m}$. Note that the person's weight (which we could find using $\Sigma F_y = 0$) does not enter into the calculation because I chose the "pivot" at the CM.

Problem 12.4 A stepladder weighs 60 N and each half is 2.0 m long. A brace 1.0 m long (of neg-ligible weight) is connected 0.50 m from the end of each half of the ladder. Assume no friction on the floor. What is the tension in the brace?



Solution Draw the force diagram for one-half of the ladder. Calculate the torque about a pivot at the top of the ladder. F is the force due to the other half. Note that the weight of half the ladder is 30 N. From $\Sigma F_y = 0, F_N - 30 = 0$, and $F_N = 30\,\text{N}$. From the drawing I see that

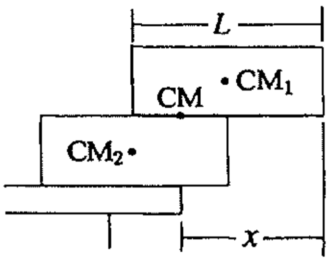


$$\sin\theta = \frac{0.5}{1.5} \qquad \theta = 19.5°$$

From $\Sigma\,\tau = 0$ about P,

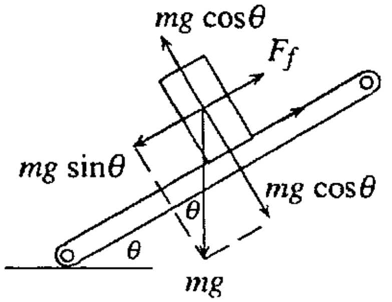
$$F_N(2\sin\theta) - (30)(1\sin\theta) - T(1.5\cos\theta) = 0$$
$$(30)(2)(\sin 19.5°) - (30)(\sin 19.5°) - 1.5T\cos 19.5° = 0 \qquad T = 7.1\,\text{N}$$

Problem 12.5 Two identical bricks, each of length L, are stacked on a table so that the top brick extends as far as possible from the edge of the table. Determine this distance.



Solution First consider the placement of only the top brick on the bottom brick. In order for the top brick not to tip, its CM (CM_1) must be just above the edge of the lower brick; that is, it must extend a distance $L/2$ beyond the lower brick edge. Now the CM of the combination of the two bricks must lie directly above the table edge. This CM is halfway between the CMs for the two identical bricks, that is, a distance $L/4$ from the outer end of the lower brick. Thus $x = 3L/4$.

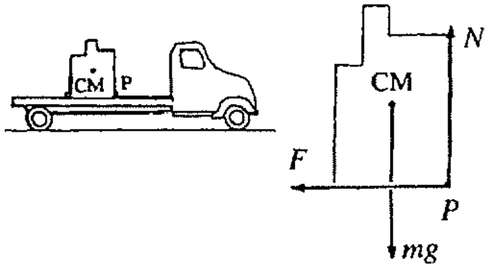
Problem 12.6 A large number of packing crates, each with a base 0.60 m × 0.60 m and 1.20 m tall, is to be loaded by a conveyer belt moving at constant speed and inclined at angle θ above horizontal. The CM of a crate is at its geometrical center. The coefficient of friction between the belt and a crate is 0.40. As θ is increased, the crate will either tip over or begin to slip. Determine the critical angles for slipping and for tipping. Which will occur first?



Solution Draw the force diagram. For tipping, the CM must be just to the left of the lower edge of the crate, so $\tan\theta_t = 0.60/1.20$, and $\theta_t = 26.6°$. For slipping, $F_f = mg\sin\theta_s$, and $F_f = \mu N = \mu mg\cos\theta_s$. So $mg\sin\theta_s = \mu mg\cos\theta_s$, $\tan\theta_s = \mu$, and $\theta_s = 21.8°$. Thus the crate will slip before it tips over as θ is increased.

Problem 12.7 A large piece of machinery is carried on a flatbed truck. Cleats keep the machinery from sliding, but only gravity keeps it from tipping. Its CM is 0.80 m above the truck bed and its base is 1.10 m from front to back. What is the shortest distance in which the truck can stop when traveling 10 m/s if the load is not to tip over?

Solution When the load is just about to tip, it will do so about the front edge at point P. Thus the upward normal force of the bed is applied there, with $N = mg$. The backward force for decelerating the load is also applied at this point. If the load is not to tip, the torque about the CM must be zero. $v^2 = v_0^2 - 2ax = 0$ to stop, and $a = F/m$, so $x = mv_0^2/2F$. For $\tau = 0$, $0.55\,mg - 0.8F = 0$. Thus F $= 0.69\,mg$.



$$x = \frac{mv_0^2}{2F} = x = \frac{mv_0^2}{2(0.69\,mg)} = \frac{(10\,\mathrm{m/s})^2}{2(0.69)(9.8\,\mathrm{m/s}^2)} = 7.42\,\mathrm{m}$$

12.2 ELASTICITY

Real materials are not perfectly rigid. When subjected to forces, they deform. For example, suppose one leg of a table is slightly longer than the others. The table will wobble, but if we put enough weight on the table, the longer leg will compress until the table is steady. However, using the methods we have examined so far, we cannot determine the exact force exerted by each leg. Such problems are indeterminate using our previous approach. A solution to finding the forces requires knowing something about the elastic properties of the leg material. If a substance deforms when subjected to a force, but returns to its initial shape when the force is removed, the substance is **elastic.**

Consider a cylinder of material of length L and cross-sectional area A. If a force F is applied along the axis of the cylinder, and this causes a change in length ΔL of the cylinder, then we define the **stress** and **strain** as

$$\text{Stress} \equiv \frac{F}{A} \tag{12.3}$$

$$\text{Strain} \equiv \frac{\Delta L}{L} \tag{12.4}$$

In a weak material, a small stress produces a large strain. For sufficiently small stresses, stress and strain are proportional. The constant of proportionality depends on the kind of material and on the nature of the deformation. The ratio of stress to strain is the **elastic modulus.**

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}} \tag{12.5}$$

Suppose you pull or push on a cylinder of length L and cross-sectional area A with a force F directed along the axis. The material is subject to a **tensile stress. Young's modulus** is defined as

$$Y = \frac{\text{tensile strength}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L} \tag{12.6}$$

If the stress exceeds the **elastic limit,** the material does not return to its original shape when the stress is removed.

The shear modulus measures a material's ability to resist changes in its shape. Suppose a piece of material in the form of a rectangular block (like a brick) has one face fixed and a force F applied to the opposite face, of area A. Imagine F applied parallel to the face, like the friction force (see Figure 12.1). If the two faces are separated by distance h, and the sheared face moves Δx, the **shear modulus** is

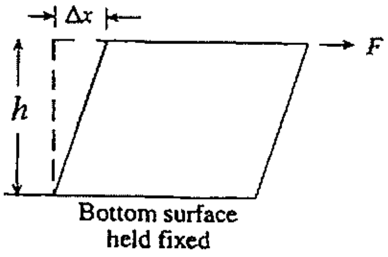


Figure 12.1

$$S = \frac{\text{shear stress}}{\text{sheer strain}} = \frac{F/A}{\Delta x/h} \tag{12.7}$$

When an object is subjected to a force from all sides, it is subject to a **pressure** P. If a force F acts perpendicular to an area A, the pressure exerted is $P = F/A$. This situation arises when an object is immersed in a fluid like air or water. When pressed from all sides, the volume V of an object will change by ΔV. The tendency for this is measured by the **bulk modulus** B defined by

$$B = \frac{\text{volume stress}}{\text{volume strain}} = -\frac{F/A}{\Delta V/V} = -\frac{P}{\Delta V/V} \tag{12.8}$$

The negative sign is inserted so that B is a positive number because ΔV is negative due to a positive pressure. In some tables of data the inverse of B, called the **compressibility**, is tabulated. A large bulk modulus means it is difficult to compress the material, whereas a large compressibility means it is easy to compress the material.

Problem 12.8 A steel beam used in the construction of a bridge is 10.2 m long with a cross-sectional area of 0.12 m^2. It is mounted between two concrete abutments with no room for expansion. When the temperature rises 10° C, such a beam will expand in length by 1.2 mm if it is free to do so. What force must be exerted by the concrete to keep this expansion from happening? Young's modulus for steel is 2.0×10^{11} N/m^2.

Solution Imagine that the steel expands and that then the concrete exerts a compressional force to return it to its original length. From Eq. 12.6,

$$F = Y(\tfrac{\Delta L}{L})A = (2 \times 10^{11}\ \mathrm{N/m^2})(\tfrac{1.2\times 10^{-3}\ \mathrm{m}}{10.2\ \mathrm{m}})(0.12\ \mathrm{m^2}) = 2.8 \times 10^6\ \mathrm{N}$$

This force could well crack the concrete. The forces involved in thermal expansion can be huge, which is why it is necessary to leave expansion space in joints in large structures like bridges and buildings.

Problem 12.9 A cube of Jell-O 6 cm on a side sits on your plate. You exert a horizontal force of 0.20 N on the top surface parallel to the surface and observe a sideways displacement of 5 mm. What is the shear modulus of the Jell-O?

Solution From Eq. 12.7,

$$S = \frac{F/A}{\Delta x/h} = \frac{Fh}{A\Delta x} = \frac{Fh}{h^2\Delta x} = \frac{F}{h\Delta x} = \frac{0.20N}{(0.060\ \mathrm{m})(0.005\ \mathrm{m})} = 670\ \mathrm{N/m^2}$$

Problem 12.10 When an object is submerged in the ocean to a depth of 3000 m, the pressure increases by about 3030 N/m^2. By how much does a piece of aluminum of volume 0.30 m^3 decrease in volume when lowered to this depth? The bulk modulus of aluminum is 7×10^{10} N/m^2.

Solution From Eq. 12.8,

$$\Delta v = -\frac{PV}{B} = -\frac{(3030\ \mathrm{N/m^2})(0.30\ \mathrm{m^3})}{7 \times 10^{10}\ \mathrm{N/m^2}} = -1.2 \times 10^{-8}\ \mathrm{m^3} = -12\ \mathrm{mm^3}$$

12.3 SUMMARY OF KEY EQUATIONS

Rotational equilibrium: $\sum \mathbf{F} = 0 \qquad \sum \vec{\tau} = 0$

Young's modulus: $Y = \dfrac{\text{tensile strength}}{\text{tensile strain}} = \dfrac{F/A}{\Delta L/L}$

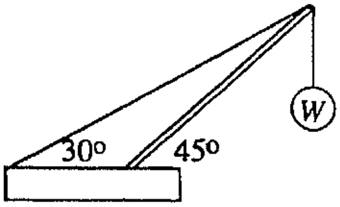
Shear modulus: $S = \dfrac{\text{shear stress}}{\text{shear strain}} = \dfrac{F/A}{\Delta x/h}$

Bulk modulus: $B = \dfrac{\text{volume stress}}{\text{volume strain}} = -\dfrac{F/A}{\Delta V/V} = -\dfrac{P}{\Delta V/V}$

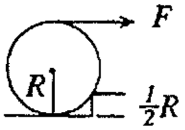
Stress $\equiv \frac{F}{A}$ Strain $\equiv \frac{\Delta L}{L}$

Supplementary Problems

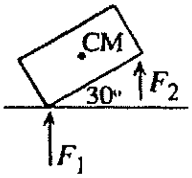
SP 12.1 In the crane here the boom is 3.2 m long and weighs 1200 N. The cable can support a tension of 10,000 N. The weight is attached 0.5 m from the end of the boom. What maximum weight can be lifted?



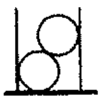
SP 12.2 What horizontal force applied as shown here is required to pull a wheel of weight W and radius R over a curb of height $h = R/2$?



SP 12.3 Two people carry a refrigerator of weight 800 N up a ramp inclined at 30° above horizontal. Each exerts a vertical force at a corner. The CM of the refrigerator is at its center. Its dimensions in the drawing are 0.72 m × 1.8 m. What force does each person exert?



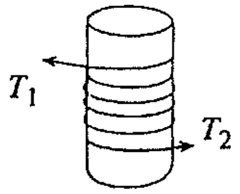
SP 12.4 A cylindrical shell of weight W and diameter $3R/2$ is placed upright on a horizontal surface. Two spheres, each of weight w and radius R, are placed in the cylinder. What are the contact forces exerted by the spheres on the cylinder? What is the maximum value of w for which the cylinder will not tip over?



SP 12.5 A plank 7.2 m long of mass 20 kg extends 2.4 m beyond the edge of a cliff. How far beyond the edge of the cliff can a 15-kg child walk before the plank tips?

SP 12.6 A string is wrapped many times around a cylinder, covering its surface. The cylinder is placed on a plane inclined at angle θ above horizontal. The end of the string is directed horizontally, where it is attached, thereby holding the cylinder in place. The coefficient of friction between the string-covered cylinder and the plane is μ. Determine the maximum value of θ for which the cylinder will remain in place and not start to move down the plane with the string unwinding.

SP 12.7 To moor a ship, a sailor wraps a rope around a bollard (a cylindrical post). By pulling with a small force T_1, he can exert a much larger tension T_2 on the end of the rope attached to the ship because of the friction between the rope and the bollard. The coefficient of friction between the rope and the bollard is $\mu = 0.2$. If the sailor pulls with 400 N, how many turns are needed if he is to exert a force of 24,000 N on the ship?



SP 12.8 The separation between the front and back axles of a bicycle is 1.14 m, and the CM of the bike plus rider is 1.20 m above the ground. The coefficient of friction between the tires and the roadway is 0.60. Determine the braking deceleration when (a) both brakes are applied, (b) only the front brake is applied, and (c) only the back brake is applied.

SP 12.9 A ladder 6 m long weighs 120 N. It leans against a smooth wall (negligible friction), making an angle of 50° with horizontal. The coefficient of friction with the floor is 0.5. How far up the ladder can an 800-N worker climb before the ladder starts to slip?

Solutions to Supplementary Problems

SP 12.1 Calculate the torque about the base of the boom. The cable is one side of a triangle with angles 30°, 135°, and 15°.

$$(10,000\,\mathrm{N})(3.2 \sin 15°) - w(2.7 \cos 45°) = 0, \qquad w = 4340\,\mathrm{N}$$

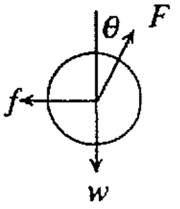
SP 12.2 The torque due to F about the contact point must balance the torque due to gravity acting at the center of the sphere.

$$w\left(\frac{\sqrt{3}}{2}R\right) - F\left(\frac{3R}{4}\right) = 0, \qquad F = \frac{2\sqrt{3}}{3}W$$

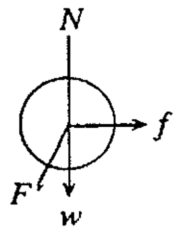
SP 12.3 $F_1 + F_2 = 800\,\mathrm{N}$, and the torque about the point of application of F_1 is zero.

$$(F_2)(1.8 \cos 30°) - (800\,\mathrm{N})(0.9 \cos 30° - 0.36 \sin 30°) = 0, \qquad F_2 = 308\,\mathrm{N} \qquad F_1 = 492\,\mathrm{N}$$

SP 12.4 Draw the force diagram for each sphere. The upward normal force of the table must support the weight of the two spheres, so $N = 2w$. Since the cylinder is not moving sideways, the two horizontal contact forces f must be equal. The contact force between the spheres is F, directed at an angle θ to vertical, where $\sin\theta = 0.5$. Applying $\Sigma F_y = 0$ to the upper sphere yields $F \cos\theta - w = 0$, so $F = 1.15w$. Applying $\Sigma F_x = 0$ yields $F \sin\theta - f = 0$, so $f = 0.58w$ If w is increased, the cylinder will tend to tip about point P. The weight of the cylinder acts downward from its CM on its

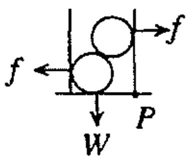


Upper Sphere



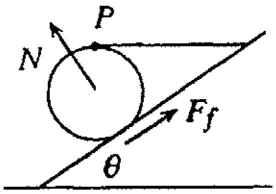
Lower Sphere

axis with a lever arm of $3R/4$ about point P. The lower contact force has lever arm R about P, and the upper contact force has lever arm $R + 2R\cos\theta$ about P. Thus for no tipping, $W(3R/4) + fR - f(R + 2R\cos\theta) = 0$,where $f = 0.58w$, and thus $w = 0.75W$.



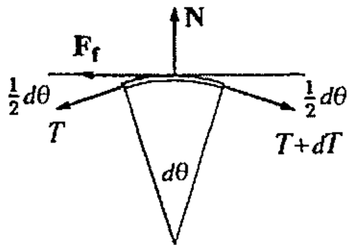
SP 12.5 The CM of the plank (at its center) is 1.2 m from the edge. The torque due to it must balance the torque due to the child, who has walked a distance x beyond the edge. Here $g = 9.8\,\text{m/s}^2$. $20g(1.2\ \text{m}) - 15gx = 0$, so $x = 1.6\,\text{m}$.

SP 12.6 If the string is not to unwind, the torque about point P must vanish. The friction force creates a counterclockwise torque with lever arm $R + R\cos\theta$, and the normal force creates a clockwise torque about P with lever arm $R\sin\theta$. Thus $F_f(R + R\cos\theta) - NR\sin\theta = 0$. Also $F_f = \mu N$. Thus



$$\mu = \frac{\sin\theta}{1 + \cos\theta}$$

SP 12.7 Look at a small segment of rope that subtends a small angle $d\theta$. Because of friction the tension at one end is T and at the other end slightly larger, $T + dT$. Apply $\Sigma F_y = 0$.



$$N - T\sin\tfrac{d\theta}{2} - (T + dT)\sin\tfrac{d\theta}{2} = 0$$

For $d\theta \ll 1$,

$$\sin\tfrac{d\theta}{2} \simeq \tfrac{d\theta}{2}$$

Neglect the very small term $2dT\sin d\theta/2$. Thus $N = 2T\ d\theta/2 \simeq T\ d\theta$. The friction force is $F_f = \mu N = \mu T\ d\theta$. Apply $\Sigma F_x = 0$:

$$(T + dT)\cos\tfrac{d\theta}{2} - F_f - T\cos\tfrac{d\theta}{2} = 0$$

$$\cos\tfrac{d\theta}{2} \simeq 1 \quad \text{if}\ \tfrac{d\theta}{2} \ll 1, \quad \text{so}\ dT = F_f = \mu T\ d\theta \quad \text{or} \quad \tfrac{dT}{T} = \mu d\theta$$

If the tensions at the two ends are T_1 and T_2, then

$$\int_{T_1}^{T_2}\tfrac{dT}{T} = \mu\int_{\theta_1}^{\theta_2} d\theta \quad \text{or} \quad \ln\tfrac{T_2}{T_1} = \mu(\theta_2 - \theta_1)$$

$$\theta_2 - \theta_1 = \tfrac{1}{\mu}\ln\tfrac{T_2}{T_1} = \tfrac{1}{0.4}\ln\tfrac{24{,}000}{400} = 10.2\ \text{rad} = \tfrac{10.2}{2\pi}\ \text{rev} \qquad \theta_2 - \theta_1 = 1.63\ \text{revs of rope}$$

SP 12.8 If normal forces on front and back wheels are N_1 and N_2, then $N_1 + N_2 = mg$.

$$L = 3.2\ \text{m} \qquad h = 0.60\ \text{m} \qquad \mu = 0.80$$

(a) Torque about CM is zero.

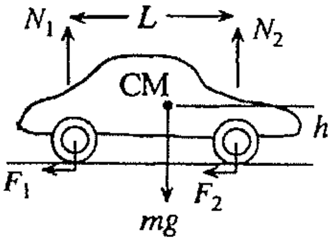
$$N_2\left(\tfrac{L}{2}\right) - N_1\left(\tfrac{L}{2}\right) - F_1h - F_2h = 0$$

$$F_1 = \mu N_1 \qquad F_2 = \mu N_2$$

Solve by finding $N_1 = 0.35\,mg$ and $N_2 = 0.65\,mg$.

$$F_{\text{net}} = ma = F_1 + F_2 = \mu N_1 + \mu N_2$$

so $a = \frac{F_{\text{net}}}{m} = \frac{\mu N_1 + \mu N_2}{m} = 0.8\,g = 7.84$ m/s^2



(b) Torque about CM is zero: $N_2(L/2) - N_1(L/2) - \mu N_2 h = 0$. Solve by finding

$$N_1 = 0.41\,mg \qquad N_2 = 0.59\,mg \qquad a = \frac{\mu N_2}{m} = 4.63\text{ m/s}^2$$

(c) Torque about CM is zero: $N_2(L/2) - N_1(L/2) - \mu N_1 h = 0$ Solve:

$$N_1 = 0.43\,mg \qquad N_2 = 0.57\,mg \qquad a = \frac{\mu N_1}{m} = 3.37\text{ m/s}^2$$

SP 12.9

$$\sum F_y = 0 \qquad N_1 - W_L - W = 0$$

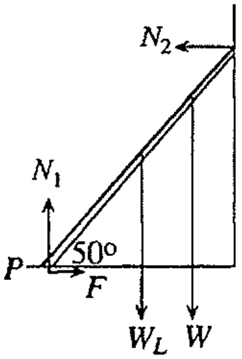
$$N_1 - 120 - 800 = 0 \qquad N_1 = 920N$$

$$\sum F_x = 0 \qquad F - N_2 = 0$$

$$F = \mu N_1 = (0.5)(920N) = 460N, \quad \text{so } N_2 = 460N$$

Torque about base of ladder, point P, is zero. The worker climbs distance x up ladder.

$$N_2(L \sin 50°) - W_L(\tfrac{L}{2} \cos 50°) - Wx \cos 50° = 0 \qquad \text{Solve } x = 3.67\text{ m}$$



Chapter 13

Oscillations

When a particle or a system repeats the same motion again and again at regular time intervals, the motion is **periodic.** We also use the word **oscillatory** to describe periodic motion, usually in the context of a vibrating object or small system, such as a mass on a spring or a violin string or an electric circuit, whereas we describe the motion of a planet around the sun as "periodic." Many kinds of oscillatory behavior are analogous to the motion of a mass attached to a spring, and it is important to understand thoroughly this system.

13.1 SIMPLE HARMONIC MOTION

Consider a mass attached to a spring with spring constant k. The spring exerts a force $-kx$, where x is the displacement of the mass from equilibrium. The law $F = ma$ is thus

$$\boxed{F = m\frac{d^2x}{dt^2} = -kx} \tag{13.1}$$

We can integrate this equation twice to obtain the solution, which is

$$\boxed{x = A\cos(\omega t + \theta)} \tag{13.2}$$

where

$$\boxed{\omega = 2\pi f = \sqrt{\frac{k}{m}}} \tag{13.3}$$

Equation 13.2 describes **simple harmonic motion**. A is the **amplitude**. f is the frequency in vibrations per second. One vibration per second is called **1 hertz** (Hz). ω is the angular frequency, in radians per second. θ is the **phase constant**. The quantity in parenthesis $(\omega t + \theta)$ is the **phase** of the oscillation. A and θ are determined by the initial conditions of the problem. All of these terms are very important. You can confirm that Eq. 13.2 is indeed the solution of Eq. 13.1 by differentiating. Thus

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \theta) \tag{13.4}$$

$$a = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \theta) \tag{13.5}$$

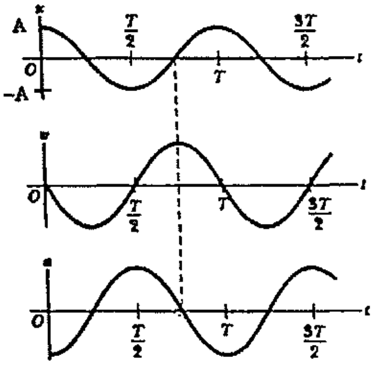


Figure 13.1

or $a = -\omega^2 x = -\frac{k}{m}x$, so $F = ma = -kx$

Graphs of displacement, velocity, and acceleration are shown in Figure 13.1 for the case $\theta = 0$. The time repeat interval T is the **period**.

$$\boxed{T = \frac{1}{f}} \qquad (13.6)$$

Note that the displacement curve and the acceleration curve are similar, but a is shifted by $T/2$ from x. a is "out of phase" with x, or "180° out of phase" with x. Similarly, v is 90° out of phase with x, and v leads x by 90° in phase, because the peak of v occurs at an earlier time than the nearest peak of x.

The maximum value of x is A, the maximum value of v is $A\omega$, and the maximum value of a is $A\omega^2$. When the displacement is large, the mass stops and $v = 0$. At this point the spring is fully stretched and F and a are both large (but negative).

If the mass on a spring is displaced a distance x_0 and released ($v = 0$ at $t = 0$), then $\theta = 0$ and $x_0 = A$. This is the case we will most often encounter.

Problem 13.1 A beam oscillates according to $x = (0.002\,\text{m})\cos(\pi t)$. What are the amplitude, maximum velocity, maximum acceleration, frequency, and period?

Solution By comparison with Eqs. 13.2, 13.4, and 13.5, $A = 0.002\,\text{m}$, $\omega = \pi$, $v_{\max} = A\omega = 0.002\pi\,\text{m/s}$, $a_{\max} = A\omega^2 = 0.002\pi^2\,\text{m/s}^2$, $f = \omega/2\pi = 0.5\,\text{Hz}$, and $T = 1/f = 2.0\,\text{s}$.

We sometimes encounter a mass hanging from a spring. The equilibrium position thus corresponds to the spring being initially stretched somewhat. However, we can show that the same equations as above still apply.

Problem 13.2 Determine the motion of a mass m hung from a spring with spring constant k.

Solution Let $y = 0$ at the end of the unstretched spring (that is, before the mass is attached), taking y as positive downward. With the mass attached, $F = ma$ becomes

$$m\frac{d^2y}{dt^2} = mg = ky$$

The mass stretched the spring a distance of $\Delta y = mg/k$ to the new equilibrium point, so transform to a new variable y', where $y' = y - mg/k$. Now $F = ma$ becomes

$$m\frac{d^2y'}{dt^2} = -ky'$$

This is the same equation as Eq. 13.1, and once again the solution is SHM. The only change is that the equilibrium point has been shifted downward by the action of the gravity force.

Problem 13.3 When a mass of 0.050 kg is suspended from a spring, it stretches the spring by 0.012 m. If now the mass is displaced slightly and allowed to oscillate, what will be its frequency?

Solution The spring constant is $k = mg/x$ (k is always positive, and here x stands for the amount of stretch, a positive distance).

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{mg}{mx}} = \frac{1}{2\pi}\sqrt{\frac{g}{x}} = \frac{1}{2\pi}\sqrt{\frac{9.8}{0.012}}\,\text{s}^{-1} = 28.6\,\text{Hz}$$

Problem 13.4 When a mass is attached to a spring, it is observed to oscillate at 10 Hz. Suppose the spring is cut in half and again the same mass is attached. At what frequency will the mass now oscillate?

Solution Suppose the force mg stretched the whole spring a distance x. This means half of the spring stretched a distance $x/2$ due to the force mg. Thus when the spring is cut in half, its new spring constant is $k' = 2mg/x = 2k$. Thus the new frequency is

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \sqrt{2}\,f_1 = \sqrt{2}\,(10\,\text{Hz}) = 14\,\text{Hz}$$

13.2 ENERGY AND SIMPLE HARMONIC MOTION

We saw previously (Eq. 8.10) that the potential energy of a spring is

$$\boxed{U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t + \theta)} \tag{13.7}$$

The kinetic energy is

$$\boxed{K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \theta)} \tag{13.8}$$

The total energy of the simple harmonic oscillator is

$$E = K + U = \tfrac{1}{2}kA^2[\sin^2(\omega t + \theta) + \cos^2(\omega t + \theta)] = \tfrac{1}{2}kA^2 \tag{13.9}$$

since $\sin^2(\omega t + \theta) + \cos^2(\omega t + \theta) = 1$.

This result makes sense, since when the spring is fully stretched, $v = 0$, $x = A$, and the energy is all potential energy. When the mass passes through the equilibrium position ($x = 0$), the potential energy is zero and the energy is all kinetic. Then $E = 1/2\ mv^2_{\max}$. Since $v_{\max} = A\omega$, $E = 1/2\ mA^2\omega^2 = 1/2\ mA^2k/m = 1/2\ kA^2$, the same result as above. As the mass oscillates, its energy switches back and forth between kinetic energy and potential energy, with the sum of the two remaining constant.

Problem 13.5 A mass is attached to a spring and displaced and then released from rest. Determine the time when the KE and PE are first equal.

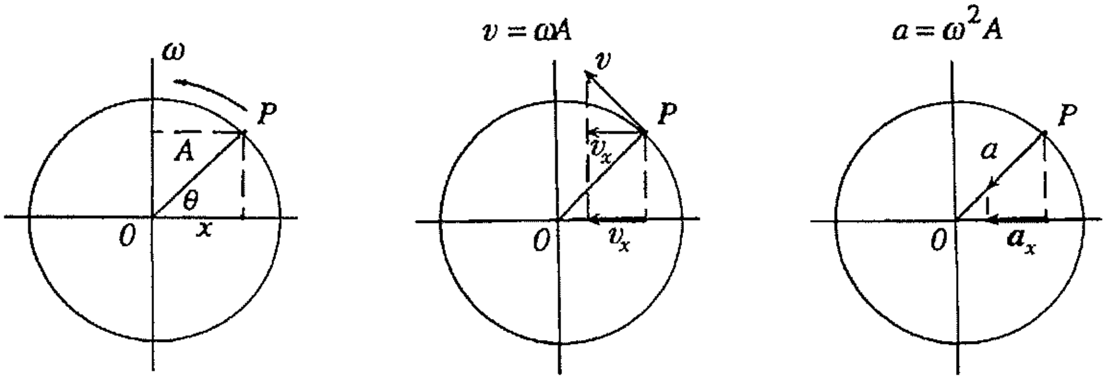
Solution $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$, so $m(-A\omega \sin \omega t)^2 = k(A \cos \omega t)^2$

$$\tan^2 \omega t = \frac{k}{m\omega^2}, \text{ and } \omega^2 = \frac{k}{m}, \text{ so } \tan^2 \omega t = 1$$

$$\omega t = \frac{\pi}{4}, \qquad t = \frac{\pi}{4\omega} = \frac{T}{8}$$

13.3 SHM AND CIRCULAR MOTION

Simple harmonic motion (SHM) can be related to circular motion in the following way. Imagine a peg P attached to a wheel oriented with its axis perpendicular to the plane of Figure 13.2. The peg is a distance A from the axis, and the wheel rotates with constant angular velocity ω. Light is directed down from above so that the peg casts a shadow on the horizontal plane (the x axis in Figure 13.2*a*).



(a) (b) (c)

Figure 13.2

At $t = 0$, the peg is all the way to the right and the shadow is at $x = A$. At a later time the position of the shadow is $x = A\cos\theta = A\cos\omega t$. The tangential velocity of the peg is of magnitude $A\omega$, and its projection on the x axis is $v = -A\omega \sin \omega t$ as shown in Figure 13.2*b*. The acceleration of the peg (centripetal) is $A\omega^2$ directed as shown in Figure 13.2*c*. The projection of the acceleration on the x axis is $a = -A\omega^2 \cos t$. Thus we see that the x position of the shadow exhibits simple harmonic motion since the equations for x, v, and a are the same as obtained above. If instead of setting $t = 0$ when the shadow was all the way to the right, we had chosen a different starting point with $\omega t = \theta$, our equations would have included the phase angle θ.

From the above discussion you can see why ω is sometimes called the *angular velocity*, as well as the *angular frequency*.

13.4 THE PENDULUM

A simple pendulum consists of a mass m suspended from a light string of length L as in Figure 13.3. If the linear displacement s is measured along the arc, $F = ma$ becomes

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad (13.10)$$

Since $s = L\theta$, this may be written

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \simeq -\frac{g}{L}\theta \qquad (13.11)$$

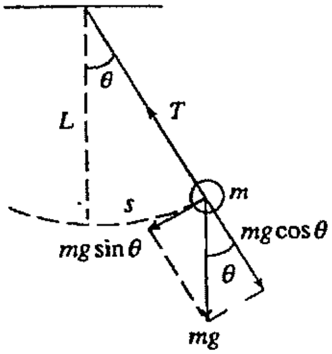


Figure 13.3

where I made the approximation $\sin\theta \simeq \theta$ for small angles. This is of the same form as Eq. 13.1, so the solution is

$$\theta = \theta_0 \cos(\omega t + \theta) \qquad (13.12)$$

θ_0 is the maximum angular displacement. The angular frequency and period are

$$\boxed{\omega = \sqrt{\frac{g}{L}} \qquad \text{and} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}} \qquad (13.13)$$

Somewhat surprisingly, the frequency depends only on g and on the length of the pendulum, not on its mass or on θ_0 (as long as the oscillations are small).

Problem 13.6 The period of a simple pendulum is 2 s. What will be the period if the mass and the length of the pendulum string are both doubled?

Solution Changing the mass will have no effect. From Eq. 13.13,

$$\frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}}, \qquad \text{so } T_2 = \sqrt{\frac{L_2}{L_1}}T_1 = \sqrt{2}T_1 = 2.8\text{ s}$$

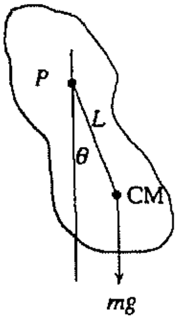


Figure 13.4

The **physical** (or **compound**) **pendulum** is a pendulum consisting of an extended rigid body pivoted at one point, P (Figure 13.4). Gravity exerts a torque that tends to restore the object to its equilibrium position (with the center of mass directly below the point of support). If the moment of inertia about the pivot P is I, the equation of motion $\tau = I\alpha$ becomes

$$-mgL\sin\theta = I\frac{d^2\theta}{dt^2} \qquad (13.14)$$

The minus sign indicates that the torque tends to decrease θ. If we again limit the oscillations to small angles, $\sin\theta \simeq \theta$. Equation 13.14 becomes

$$\frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega^2\theta \qquad (13.15)$$

This is the same as Eq. 13.1, and the solution is SHM, $\theta = \theta_0 \cos(\omega t + \theta)$. The period and angular frequency are

$$\omega = \sqrt{\frac{mgL}{I}} \qquad \text{and} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}} \qquad (13.16)$$

Problem 13.7 In normal walking, the legs of a human or animal swing more or less freely like a physical pendulum. This observation has enabled scientists to estimate the speed at which extinct creatures such as dinosaurs traveled. If a giraffe has a leg length of 1.8 m, and a step length of 1 m, what would you estimate to be the period of its leg swing? How fast would it then travel when walking?

Solution We can model the giraffe's leg as a physical pendulum of length L pivoted about one end.

$$I = \tfrac{1}{3}mL^2$$

for a rod pivoted at one end, from Table 10.2.

$$T = 2\pi\sqrt{\frac{mL^2}{3mgL}} = 2\pi\sqrt{\frac{L}{3g}} = 1.6\,\text{s} \qquad v = \frac{\text{step length}}{\text{period}} = \frac{1\,\text{m}}{1.6\,\text{s}} = 0.6\,\text{m/s}$$

Problem 13.8 A physical pendulum consists of a sphere of mass m and radius R attached to the end of a string of length L and negligible mass. What is the period of this pendulum?

Solution By the parallel-axis theorem, Eq. 10.9, the moment of inertia of the sphere about the point where the string is attached to the ceiling is

$$I = I_{\text{CM}} + m(R+L)^2 = \tfrac{2}{5}\,mR^2 + m(R+L)^2$$

From Eq. 13.16 the period is

$$T = 2\pi\sqrt{\frac{I}{mg(L+R)}} = 2\pi\sqrt{\frac{2R^2 + 5(R+L)^2}{5g(R+L)}}$$

In the limit $R \ll L$, this reduces to the period of a simple pendulum of length L. Note also that the period of this physical pendulum is longer than that of a simple pendulum of length L or of length $L + R$.

A **torsional pendulum** can be constructed by attaching a mass to a watch spring or by suspending a mass from a thin fiber that can exert a restoring torque when twisted. If the torque is proportional to the angular displacement, as is often true for small displacements, then $\tau = -\kappa\theta$, where κ is the torsional constant of the spring or fiber. Proceeding as above, we conclude that SHM results, with

$$\omega = \sqrt{\frac{\kappa}{I}} \qquad (13.17)$$

Here I is the moment of inertia of the mass attached to the spring or fiber.

13.5 DAMPED OSCILLATIONS AND FORCED OSCILLATIONS

Real oscillators experience dissipative forces such as friction that damp the motion. Frequently such damping forces may be approximated by a term $-bv$ in the force equation. In this case the equation $F = ma$ becomes

$$m\frac{d^2x}{dt^2} = -kx - bv \qquad (13.18)$$

The solution is

$$x = Ae^{-(b/2m)t}\cos(\omega t + \theta) \qquad (13.19)$$

where

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \qquad (13.20)$$

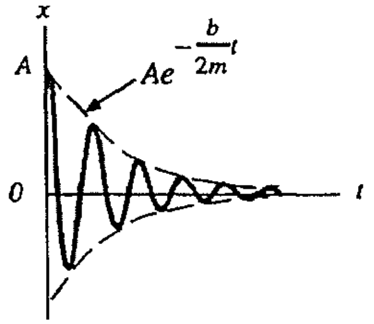


Figure 13.5

Here $\omega_0 = \sqrt{k/m}$, the "natural frequency." Figure 13.5 is a representative graph of x versus t. The exact shape of the curve depends on the size of the damping parameter b. In all cases there is exponential decay.

Frequently an oscillator is driven with a force $F_0 \cos\omega t$. When this is done, it is found that the system oscillates at the driving frequency ω, but the displacement has a difference in phase from the driving force (Figure 13.6). The amplitude of oscillation A is strongly dependent on how close ω is to the natural frequency ω_0. When $\omega = \omega_0$, the amplitude can become very large. This condition is called **resonance**. A large driving force at the resonant frequency can cause a large structure to collapse, as happened to the Tacoma Narrows bridge in 1940 due to vibrations driven by a light wind.

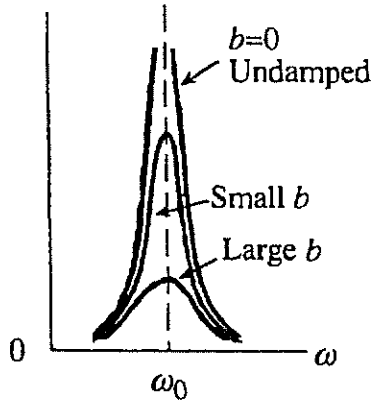


Figure 13.6

13.6 SUMMARY OF KEY EQUATIONS

Simple harmonic motion: $F = -kx$

$$x = A\cos(\omega t + \theta) \qquad v = -A\omega\sin(\omega t + \theta) \qquad a = -A\omega^2\cos(\omega t + \theta)$$

$$E = \mathrm{PE} + \mathrm{KE} = \frac{kA^2}{2} \qquad \mathrm{PE} = U = \frac{kx^2}{2} \qquad \omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$$

Pendulum: $\omega = \sqrt{\frac{g}{L}}$

Damped oscillator: $x = Ae^{-(b/2m)t}\cos(\omega t + \theta)$

Supplementary Problems

SP 13.1 A 0.20-kg mass sliding on a horizontal plane is attached to a spring, stretched 0.12 m, and released from rest. After 0.40s the speed of the mass is zero. What is the maximum speed of the mass?

SP 13.2 The piston in a car engine undergoes SHM, moving back and forth a distance of 0.084 m at 2400 rev/min. The piston mass is 1.25 kg. What are the maximum speed and acceleration of the piston? What is the maximum force on it?

SP 13.3 A spring rests on a plane inclined at angle θ above horizontal. The upper end of the spring is fixed, and the lower end is attached to a mass m that can slide on the plane without friction. What is the angular frequency for oscillations of the mass?

SP 13.4 A piston undergoes SHM with a frequency of 2 Hz. A coin rests on top of the piston. What is the maximum amplitude for which the coin will always remain in contact with the piston?

SP 13.5 When a mass m is attached to a particular spring, it oscillates with a period T. With what period will it oscillate if attached to two such springs connected side by side (in "parallel")?

SP 13.6 A car of mass 1200 kg oscillates on its springs at a frequency of 0.50 Hz with an amplitude of 0.04 m. What is the energy of this motion?

SP 13.7 A 0.24-kg block rests on a table, and a 0.12-kg block is placed on top of it. The coefficient of friction between the blocks is 0.25. The lower block is now moved back and forth horizontally in SHM with amplitude 0.05 m. What is the highest frequency for which the upper block will not slip with respect to the lower block?

SP 13.8 A mass undergoes SHM with amplitude A. What fraction of the energy is kinetic energy when $x = A/2$?

SP 13.9 A grandfather's clock is to be designed so that on every half swing a small gear is moved one notch to indicate the passage of 1 s (that is, the period of the simple pendulum is 2 s). What length pendulum is required?

SP 13.10 A student attempts to use a simple pendulum to measure g, the acceleration due to gravity. She observes that a pendulum of length 1.50 m makes 24 oscillations in 60 s. What is the value of g at her location?

SP 13.11 A disk of mass m and radius R is pivoted at a point on its perimeter and allowed to swing freely parallel to its plane. What is the period of such motion for small oscillations?

SP 13.12 An engineer wishes to determine the moment of inertia of a machine part of mass 1.20 kg about a particular axis (the X-X' axis) through the CM. He locates the CM by suspending the object motionless from several different points around its periphery. He then suspends the object from a pivot a distance 0.25 m from the CM and observes that it undergoes small oscillations with a period of 1.50 s about an axis parallel to the X-X' axis. What is the moment of inertia of the object about the X-X' axis?

SP 13.13 A damped simple harmonic oscillator is characterized by $m = 0.2$ kg, $k = 80$ N $\cdot$ m, and $b = 0.072$kg/s. What is the period? How long does it take for the amplitude to decrease to half its original value?

Solutions to Supplementary Problems

SP 13.1 $T = 0.8\,\text{s}, \quad A = 0.12\,\text{m}, \quad v_{\max} = A\omega = A\left(\frac{2\pi}{T}\right) = 0.94\,\text{m/s}$

SP 13.2 $f = \frac{2400}{60}\,\text{s}^{-1} = 40\,\text{s}^{-1}, \quad a_{\max} = A\omega^2 = (0.042\,\text{m})(2\pi)^2(40\text{s}^{-1})^2 = 2650\,\text{m/s}^2$

$$v_{\max} = A\omega = 10.5\,\text{m/s}, \qquad F_{\max} = ma_{\max} = (1.25\,\text{kg})(2650\,\text{m/s}^2), \qquad F_{\max} = 3310\,\text{N}$$

SP 13.3 The force along the plane is $-ks + mg\sin\theta$, measuring displacements as positive down the plane, where $s = 0$ at equilibrium. The equation of motion is $m\,(d^2s/dt^2) = -ks + mg\sin\theta$. Let

$$s' = s - mg\sin\theta, \quad \text{and} \quad m\frac{d^2s'}{dt^2} = -ks', \quad \text{so}\;\omega = \sqrt{\frac{k}{m}}$$

SP 13.4 If the coin leaves the piston, it will do so just as the piston is starting its downward motion, where its acceleration is a maximum. This acceleration should not exceed g if the coin is not to lose contact.

$$a_{\max} = A\omega^2 = g, \qquad A = \frac{g}{\omega^2} = \frac{9.8\,\text{m/s}^2}{(2\pi)^2(2\,\text{s}^{-1})^2} = 0.062\,\text{m}$$

SP 13.5 If force F stretches one spring a distance x, it will require force $2F$ to stretch two parallel springs the same amount, so the effective spring constant is

$$k' = 2k \qquad \text{and} \qquad T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}} = \frac{1}{\sqrt{2}}T$$

SP 13.6 From Eq. 13.9, $E = 1/2\ kA^2$, and $\omega^2 = k/m$. Thus

$$E = \tfrac{1}{2}m\omega^2A^2 = \tfrac{1}{2}(1200\ \text{kg})(2\pi)^2(0.50\text{s}^{-1})^2(0.04\ \text{m})^2 \qquad E = 9.5\ \text{J}$$

SP 13.7 Friction acts to move the upper block, so for that block, $m_1a_1 = F_f = \mu m_1 g$; thus $a_1 = \mu g$. The acceleration of the lower block should not exceed this if the upper block is not to slip; therefore,

$$a_{\max} = A\omega^2 = \mu g, \qquad \omega^2 = \frac{\mu g}{A} = \frac{(0.25)(9.8\ \text{m/s}^2)}{(0.05\text{m})} \qquad \omega = 7\ \text{rad/s}, \qquad f = \frac{\omega}{2\pi} = 1.1\ \text{Hz}$$

SP 13.8 From Eq. 13.9, the total energy is $E = 1/2\ kA^2$. When $x = 1/2\ A$, PE $= 1/2\ kx^2 = 1/8\ kA^2$. PE + KE $= E$, so $1/8\ kA^2$ + KE $= 1/2\ kA^2$, and KE $= 3/8\ kA^2 = 3/4\ E$.

SP 13.9 $T = 2\pi\sqrt{\frac{L}{g}}$, $\qquad L = \left(\frac{T}{2\pi}\right)^2 g = \left(\frac{2\text{s}}{2\pi}\right)^2(9.8\text{m/s}^2) = 0.99\ \text{m}$

SP 13.10 $T = \frac{60\,\text{s}}{24} = 2.5\,\text{s}$, $\qquad T = 2\pi\sqrt{\frac{L}{g}}$, $\qquad g = L\left(\frac{2\pi}{T}\right)^2 = 9.47\ \text{m/s}^2$

SP 13.11 From the parallel-axis theorem, Eq. 10.9, $I = I_{\text{CM}} + mR^2$. From Table 10.2, $I_{\text{CM}} = 1/2\ mR^2$, so $I = 3/2\ mR^2$. From Eq. 13.16,

$$T = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{3/2\ \text{m}R^2}{mgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

SP 13.12 $T = 2\pi\sqrt{\frac{I}{mgL}}$, $\quad I = mgL\left(\frac{T}{2\pi}\right)^2 = (1.2\ \text{kg})(9.8\ \text{m/s}^2)(0.25\text{m})\left(\frac{1.5\text{s}}{2\pi}\right)^2 = 0.17\ \text{kg}\cdot\text{m}^2$

$$I = I_{\text{CM}} + mL^2, \quad \text{so } I_{\text{CM}} = I - mL^2 = 0.17\text{kg}\cdot\text{m}^2 - (1.2\ \text{kg})(0.25\ \text{m})^2 = 0.10\ \text{kg}\cdot\text{m}^2$$

SP 13.13 From Eq. 13.18,

$$A = A_0e^{-(b/2m)t} = \tfrac{1}{2}A_0, \quad \text{so } e^{-(b/2m)t} = 2, \text{and } \ln^{-(b/2m)t} = \ln 2$$

$$\frac{bt}{2m} = \ln 2, \qquad \text{since } \ln e = 1, \qquad \text{and} \qquad t = \frac{2m\ln 2}{b} = \frac{(2)(0.2\ \text{kg})(\ln 2)}{0.072\ \text{kg/s}} = 3.9\ \text{s}$$

$$\frac{k}{m} = \frac{80\,\text{N}\cdot\text{m}}{0.20\text{kg}} = 400\text{s}^{-2}, \qquad \left(\frac{b}{2m}\right)^2 = \left(\frac{0.072\ \text{kg/s}}{(2)(0.2\ \text{kg})}\right)^2 = 0.03\ \text{s}^{-2}$$

$$\text{so } \frac{k}{m} \gg \left(\frac{b}{2m}\right)^2 \qquad \text{and} \qquad \omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \simeq \sqrt{\frac{k}{m}}\ f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 3.18\ \text{Hz}$$

Chapter 16

Waves and Sounds

A wave is a periodic disturbance that travels through space. Examples are water waves, sound waves, electromagnetic waves (for example, radio waves, microwaves, light, and x-rays), and vibrational waves on a stretched string. In quantum mechanics we encounter probability waves that tell us the likelihood of finding an electron at one place or another. We can learn about the basic properties of waves by studying the waves that propagate on a stretched string, and from there we can go on to understand other kinds of waves.

16.1 TRANSVERSE MECHANICAL WAVES

Suppose one end of a string is tied to something, and you hold the other end, pulling the string taut. If you now give a sudden jerk on your end of the string, a pulse will travel along the string, as shown in Figure 16.1. Shown there are "snapshots" of the string at successively later time intervals. The pulse travels at speed v, the **wave velocity**, in the positive x direction.

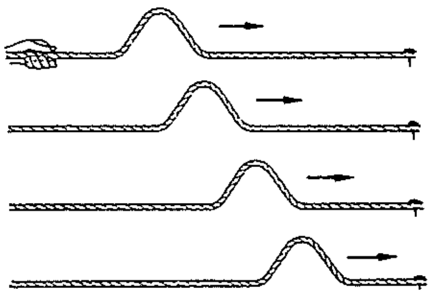


Figure 16.1

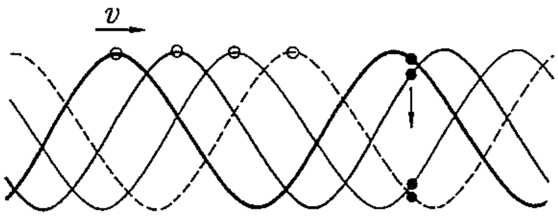


Figure 16.2

If you move your hand up and down in simple harmonic motion, you will generate a **transverse sinusoidal traveling wave** on the string, as shown in Figure 16.2. In this drawing the open circles indicate the progress of a certain feature of the wave (a peak), and the solid circles indicate the up and down motion of a piece of the string. In your mind's eye imagine a sine wave (the darkest curve) moving to the right at speed v. Its position at three subsequent times is described by the curves in the drawing. Observe that although the wave is moving to the right, no matter is moving in this direction. The particles of the string just move up and down, transverse (that is, perpendicular) to the direction of propagation of the wave. Be careful not to confuse the particle velocity dy/dt (in the y direction) with the wave velocity v (in the x direction).

Although complicated waveforms are encountered in nature, I will focus on the properties of sinusoidal waves since more complex waves can be described in terms of combinations (superpositions) of sinusoidal waves. We can obtain an equation of motion (a "wave equation") for a particle on a stretched string by applying $F = ma$ to a little piece of the string. When we do this, we find that a solution is any function whose argument is $x + vt$ or $x - vt$, that is, $f(x + vt)$ or $f(x - vt)$. The exact nature of the function $f(x - vt)$ depends on how you wiggle the end of the string. When you wiggle the end of the string in simple harmonic motion at frequency f, the transverse displacement of a piece of the string is given by

$$y(x,\, t) = A \sin \tfrac{2\pi}{\lambda}(x - vt) \qquad (16.1)$$

This describes a wave moving to the right along the x axis with speed v. The quantity $\phi = (2\pi/\lambda)\,(x - vt)$ is the **phase of the wave.** When the phase has a given value, y has a given value. Thus a constant value of y requires that the phase $(2\pi/\lambda)$ $(x - vt)$ be constant. Suppose $(2\pi/\lambda)$ $(x - vt) = \phi =$ constant. If we take the time derivative of the phase, we see that $dx/dt - v = 0$, or $v = dx/dt$, which shows that the quantity v is indeed the velocity of the wave. v is called the **phase velocity** (or wave velocity) because it describes the speed of a point of constant phase on the wave. Observe that a function of the form $f(x + vt)$, that is, with argument $(x + vt)$, instead of $(x - vt)$, describes a wave traveling toward the negative x direction because for it we find that $dx/dt = -v$, and v is a positive number.

The form of Eq. 16.1 is reminiscent of the way we described the y coordinate of a point on the rim of a rotating disk. As the disk rotated, the angle of rotation (the argument of the sine function) varied from 0 to 2π rad, or from 0 to 360°. Thus we sometimes refer to phase in terms of degrees or radians. A picture of a wave at $t = 0$ is shown in Figure 16.3. Observe that moving a distance of $\lambda/4$ in space along the x axis causes the phase to change by 90°. Moving a distance $\lambda/2$ results in a phase change of 180°.

The maximum value y reaches is the **amplitude** A in meters. The separation in space between two adjacent points with the same phase (for example, between two crests) is the **wavelength** λ, in meters, of the wave. Amplitude and wavelength are indicated in Figure 16.4. From Eq. 16.1 we see that when x increases by an amount λ, the phase increases by 2π rad, and since sin $(\phi + 2\pi) =$ sin ϕ, y has the same value at points x and at $x + \lambda$.

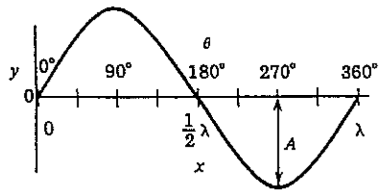


Figure 16.3

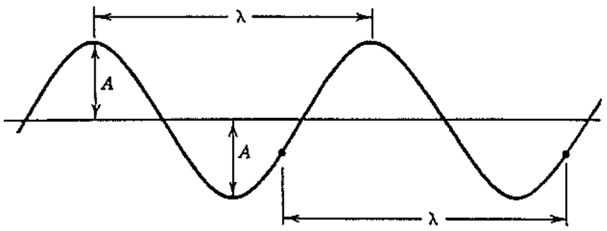


Figure 16.4

The number of crests passing a given point in space each second is the **frequency** *f.* One crest (or vibration) per second is 1 **hertz (Hz),** so 1 Hz = 1s. Consider a length of the wave of length $L = n\lambda$, where n is the number of crests in the length L. In t s it moves a distance $L = vt = n\lambda$, or $v = \lambda n/t$. But n/t is the number of crests passing a given point per second, which is the frequency f. Thus

$$\boxed{v = f\lambda = \frac{\lambda}{T}} \tag{16.2}$$

Here the time in seconds between adjacent passing crests is the **period** T = 1/f. Thus if 10 crests pass each second ($f = 10\,\mathrm{Hz}$), the period is $T = 1/f = 1/10 = 0.1\,\mathrm{s}$ = time between crests or for oscillations to repeat.

We frequently encounter the **wave number** k and the **angular frequency** ω (in radians per second) defined as follows:

$$\boxed{\omega = 2\pi f \quad \text{and} \quad k = \frac{2\pi}{\lambda}, \quad \text{so } v = f\lambda = \frac{\omega}{k}} \tag{16.3}$$

Using this notation, Eq. 16.1 can be conveniently written as

$$\boxed{y(x,\ t) = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) = A\sin\left(kx - \omega t\right)} \tag{16.4}$$

In the above I assumed the vertical displacement $y(x,\ t)$ is zero at $t = 0$ and $x = 0$. This need not be the case (we can start $t = 0$ whenever we want), so a more general form for $y(x,\ t)$ is $y = A\sin\,(kx - \omega t + \phi)$, where ϕ, the **phase constant,** is determined from the given initial conditions. If we choose the zero of time and the x-axis origin so that $\phi = 90°$, then $y = A\sin\,(kx - \omega t + 90°) = A\,\cos\,(kx - \omega t)$. In some books we see the sine function and in others the cosine function. Either form can be used.

Problem 16.1 A string wave is described by $y = 0.002\,\sin\,(0.5x - 628t)$. Determine the amplitude, frequency, period, wavelength, and velocity of the wave.

Solution From Eq. 16.4,

$$A = 0.002\,\mathrm{m} \qquad \frac{2\pi}{\lambda} = 0.5 \qquad \lambda = 12.6\,\mathrm{m}$$

$$\frac{2\pi}{T} = 628 \qquad T = 0.01\ \mathrm{s} \qquad f = \frac{1}{T} = 100\,\mathrm{Hz} \qquad v = f\lambda = 1260\,\mathrm{m/s}$$

16.2 SPEED AND ENERGY TRANSFER FOR STRING WAVES

By applying $F = ma$ to a small piece of a vibrating string, we can deduce the wave equation and the speed of the transverse waves on the string. If the tension of the string is T and the mass per unit length (the linear mass density) is μ, the wave velocity is

$$\boxed{v = \sqrt{\frac{T}{\mu}}} \qquad (16.5)$$

Although no matter is transported down the string as the wave propagates, energy is carried along by the wave with velocity v. As a piece of the string moves up and down executing simple harmonic motion, it has kinetic energy as well as potential energy (because the string is stretched like a spring). In Eq. 13.9 we saw that the total energy of a mass m that oscillates with amplitude A and angular frequency ω is $E = 1/2\ kA^2 = 1/2\ m\omega^2 A^2$, where k is the spring constant and $k = m\omega^2$. Consider a small length dx of the string. The mass of this piece is $dm = \mu dx$, where μ is the mass per unit length of the string. This infinitesimal mass of string thus has a small energy $dE = 1/2\ dm^2\ A^2 = 1/2\ \mu dx\omega^2 A^2$. As this small mass moves up and down, it pulls on the piece of string to its right and does work on it, thereby transferring energy down the string to the right as the wave moves in that direction. If energy dE is transferred in time dt, the rate of energy transfer (the power) is $P = dE/dt = 1/2\ \mu\omega^2 A^2 dx/dt$ where $dx/dt = v =$ wave velocity. Thus the power transmitted by the wave is

$$\boxed{P = \frac{1}{2}\mu v\omega^2 A^2} \qquad (16.6)$$

Problem 16.2 A string of linear mass density 480 g/m is under a tension of 48 N. A wave of frequency 200 Hz and amplitude 4.0 mm travels down the string. At what rate does the wave transport energy?

Solution $\omega = 2\pi f = 2\pi(200) = 400\pi\,\mathrm{s}^{-1} \qquad v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{48\,\mathrm{N}}{0.48\,\mathrm{kg/m}}} = 10\,\mathrm{m/s}$

$$P = \frac{1}{2}\mu v\omega^2 A^2 = (0.5)(0.480\,\mathrm{kg/m})(10\,\mathrm{m/s})\,(400\,\pi\,\mathrm{s})^2(4\times10^{-3}\,\mathrm{m})^2 = 61\,\mathrm{W}$$

Equation 16.6 proves to be qualitatively true for all kinds of waves, including electromagnetic waves such as light. The power transmitted is proportional to the wave velocity and the the square of the frequency and to the square of the amplitude.

16.3 SUPERPOSITION OF WAVES

Consider a stretched string tied at one end (Figure 16.5*a*). If you jerk the other end, a pulse will travel along the string. When the pulse reaches the tied end, it will be turned upside down (180° phase shift) and be reflected. If the far end of the string is free to move up and down (perhaps it slides on a pole, as in

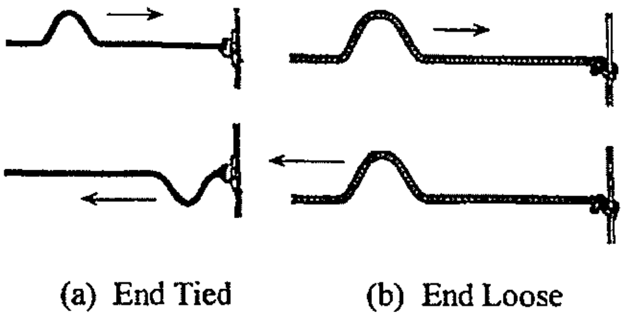
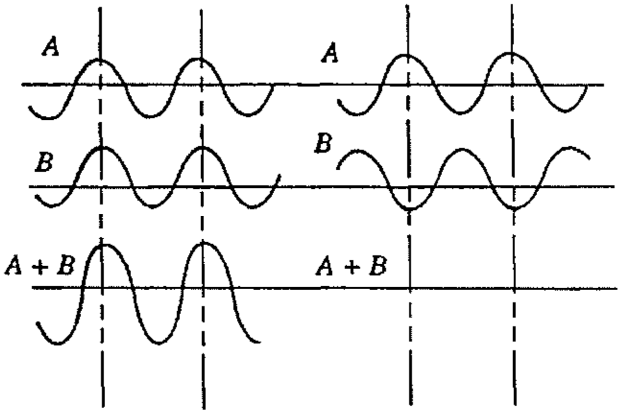

(a) End Tied (b) End Loose

Figure 16.5

Figure 16.5*b*), the reflected pulse does not turn upside down (no phase shift). Suppose two pulses are sent down the string. The first one is reflected, and on its way back, it encounters the second oncoming pulse. The two will interact (they are said to **interfere**). As they pass each other, their displacements will add. The pulses in Figure 16.5*a* are upside down from each other, and as they pass, they cancel each other out (**destructive interference**). The two pulses in Figure 16.5*b* reinforce each other (**constructive interference**). This adding together of wave displacements is called **superposition**. By adding together very many sine waves, very complicated wave forms can be constructed.

Conversely, a complicated wave pattern can be decomposed into many sine waves.

If two waves of the same velocity and wavelength are travel-ing in the same direction on a string, they will interfere. If they are in phase (Figure 16.6*a*), they interfere constructively and result in a stronger wave. If they are out of phase (Figure 16.6*b*) and have the same amplitude, they cancel each other out (destructive interference).



(a) Reinforcement (b) Cancellation

Figure 16.6

16.4 STANDING WAVES

Consider two sinusoidal traveling waves with the same amplitude and wavelength moving in opposite directions on a string. The resultant combination for the two waves is obtained by superposition; thus

$$y(x,\ t) = y_1(x,\ t) + y_2(x,\ t) = A\sin(kx - \omega t) + A\sin(kx + \omega t) \qquad (16.7)$$

Use a trig identity to simplify this: $\sin\alpha + \sin\beta = 2\ \sin 1/2\ (\alpha+\beta)\cos 1/2\ (\alpha-\beta)$. Thus

$$\boxed{y\,(x,\ t) = (2A\sin kx)\cos\omega t} \qquad (16.8)$$

Here $y(x,\ t)$ is a **standing wave**. We think of the magnitude of the quantity $2A\sin kx$ as the amplitude for simple harmonic motion of a small piece of the string at the position x. A point where the amplitude of the standing wave is zero is called a **node**. A point where the amplitude is a maximum is an **antinode**.

Now consider a string of length L with both ends fixed, so $y = 0$ at $x = 0$ and at $x = L$. Imagine that one end jiggles slightly, so that a wave travels down the wave and is reflected back. These two oppositely traveling waves can interfere and set up standing waves, as illustrated in Figure 16.7. There we see that possible standing waves are those for which the length of the string is an integer multiple of one-half wavelength.

$$L = n\frac{\lambda}{2} \quad \text{or} \quad \lambda = \frac{2L}{n} \quad \text{or} \quad \boxed{f = \frac{v}{\lambda} = n\frac{v}{2L} \quad \text{where } n = 1, 2, 3, \cdots} \qquad (16.9)$$

This makes

$$\sin kx = \sin \frac{2\pi}{\lambda} = 0 \qquad \text{at } x = L$$

The standing waves described here result only when the string oscillates at frequencies given by Eq. 16.9. These are called **resonant frequencies,** and they represent oscillations of the string with large amplitude. Waves traveling with other frequencies will not set up standing waves. Instead, they will just cause the string to vibrate with very small or imperceptible oscillations. The patterns shown in Figure 16.7 are examples of **resonant modes** of the system. Structures such as bridges, buildings and freeways have many possible resonant modes. If a structure is driven at one of its resonant frequencies, large amplitude oscillations can result, and the structure may fall down.

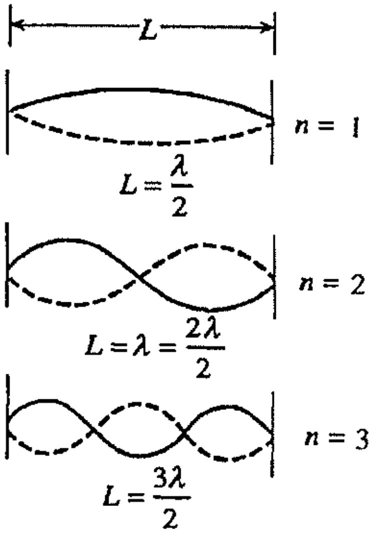


Figure 16.7

The lowest resonant frequency ($n = 1$ in Eq. 16.9) is the **fundamental frequency** or the **first harmonic** f_1. The second harmonic is the mode with $n = 2$ and frequency $f_2 = 2f_1$, and so on for the higher harmonics f_3, f_4, and so on.

Problem 16.3 The G string of a mandolin is 0.34 m long and has a linear mass density of 0.004 kg/m. The thumbscrew attached to the string is adjusted to provide a tension of 71.1 N. What then is the fundamental frequency of the string?

Solution

$$f_1 = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = \frac{1}{(2)(0.34\,\text{m})}\sqrt{\frac{71.1\,\text{N}}{0.004\,\text{kg/m}}} = 196\,\text{Hz}$$

A stringed instrument such as a guitar is tuned by adjusting the tension in a string by means of a thumbscrew. The length of the string is fixed, so adjusting the tension adjusts the fundamental frequency. Other fundamental frequencies can be achieved by shortening the string length by pressing on a fret. Finally, several strings of different mass densities are used to give a range of wave velocities, thereby providing access to a greater range of fundamental frequencies.

16.5 SOUND WAVES

A sound wave is a longitudinal pressure wave. *Longitudinal* means that the pressure variations are parallel to the direction of travel, whereas in a vibrating string waves, the variations in displacement are transverse to the wave velocity. We can envision what happens by placing a long coiled spring on a horizontal table. When one end is moved back and forth harmonically, regions of compression and rarefaction travel along the spring, as sketched in Figure 16. 8. We can derive the speed of a sound wave using $F = ma$, and the result is

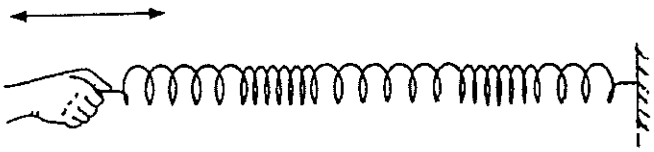


Figure 16.8

$$\boxed{v = \sqrt{\frac{B}{\rho}}} \tag{16.10}$$

Here B is the bulk modulus and ρ is the mass density of the medium in which the sound is traveling. This is similar in form to the expression for the speed of the transverse waves on a stretched spring, $v = \sqrt{T/\mu}$. In fact, the velocity of any mechanical wave is of the form $v = \sqrt{\textit{elastic property/inertial property}}$ or $\sqrt{\textit{"stiffness"/density}}$.

As for all waves, $f\lambda = v$. Representative values for sound velocities are 343 m/s in air at 20° C, 1493 m/s in water at 25° C, and 5130 m/s in iron.

Problem 16.4 For copper the bulk modulus is 14×10^{10} N/m^2 and the density is 8920 kg/m^3. What is the speed of sound in copper?

Solution $$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{14 \times 10^{10}\,\text{N/m}^2}{8920\,\text{kg/m}^3}} = 3960\text{ m/s}$$

A sound wave can transport energy since as it moves along it causes molecules to vibrate with kinetic energy. When we hear a sound wave, we detect the pitch and the loudness. The **pitch** of a sound is its **frequency**, and its **loudness** is proportional to the **power intensity** of the wave. Humans can typically hear a frequency range of 20 to 20,000 Hz (when you're 16 years old, not when you are over the hill at 25). As you get older, the high-frequency response gets worse and worse. The average power per unit area perpendicular to the direction of travel of a sound wave is the **intensity**. Humans can detect power intensities ranging from $I_0 = 10^{-12}$ W/m^2 up to about 1 W/m^2. Any higher intensities are very painful to the ear. Because sound intensities vary over such a wide range, it is convenient to use a different quantity as a measurement of intensity. A dimensionless quantity β is defined, measured in units of **decibels** (dB).

$$\boxed{\beta = 10\log_{10}\frac{I}{I_0}} \qquad \text{where } I_0 = 10^{-12}\ \text{W/m}^2 \qquad (16.11)$$

Problem 16.5 Normal conversation is carried on at about 60 dB. To what intensity level does this correspond?

Solution $60 = 10\log_{10}\frac{I}{I_0}$, so $10^6 = \frac{I}{I_0}$ and $I = 10^6 I_0 = 10^{-6}\ \text{W/m}^2$

I can deduce the power carried by a sound wave as follows: Suppose the wave is traveling along the x axis of a cylinder of material of cross-sectional area A and density ρ. A piece of mass dm occupies volume dV and is undergoing simple harmonic motion along the x axis. The average energy dE of the mass is equal to its maximum kinetic energy 1/2 $(dm)v^2_{\text{max}}$ where $v_{\text{max}} = \omega x_{\text{max}}$ is the maximum particle velocity (not the sound wave velocity) and x_{max} is the maximum amplitude of vibration. Also, $dm = \rho dV = \rho A dx$. Thus

$$dE = \tfrac{1}{2}(dm)v^2 = \tfrac{1}{2}\rho A dx\,(\omega x_{\text{max}})^2$$

Power is the rate of energy transport, so

$$P = \frac{dE}{dt} = \tfrac{1}{2}\rho A\,\omega^2 x^2_{\text{max}}\frac{dx}{dt}, \quad \text{and} \quad \frac{dx}{dt} = v \text{ is the wave velocity, so}$$

$$\boxed{P = \tfrac{1}{2}\rho A\,\omega^2 x^2_{\text{max}}\,v} \qquad (16.12)$$

Here v is the wave velocity, x_{max} is the maximum displacement, A is the cross-sectional area through which the sound is propagating, ρ is the density of the material, and $\omega = 2\pi f$, where f is the sound wave frequency. The sound intensity is defined as

$$\boxed{I = \frac{\text{power}}{\text{area}} = \tfrac{1}{2}\rho\,(\omega x_{\text{max}})^2 v} \qquad (16.13)$$

It can be shown that the variation ΔP_{max} in pressure amplitude can be expressed as

$$\Delta P_{max} = \rho\, v\, \omega\, x_{max}, \quad \text{so} \quad \boxed{I = \frac{(\Delta \text{P}_{max})^2}{2\rho v}} \qquad (16.14)$$

Problem 16.6 A source emits sound uniformly in all directions at a power level of 60 W. What is the intensity at a distance of 4 m from the source?

Solution The power is distributed over the surface area of a sphere: $A = 4\pi r^2$.

$$I = \frac{P}{4\pi r^2} = \frac{60\ \text{W}}{4\pi(4\ \text{m})^2} = 0.30\ \text{W/m}^2$$

Problem 16.7 At a distance of 5 m from a source the sound level is 90 dB. How far away has the level dropped to 50 dB?

Solution $I_1 = \frac{P}{4\pi r_1^2}$ and $I_2 = \frac{P}{4\pi r_2^2}$, so $\frac{I_2}{I_1} = \frac{r_1^2}{r_2^2}$

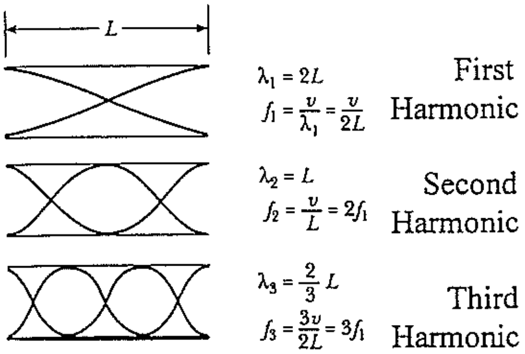
$$\beta_1 = 10 \log \frac{I_1}{I_0} = 90\,\text{dB}, \qquad \text{so } \frac{I_1}{I_0} = 10^9$$

Similarly, $$\frac{I_2}{I_0} = 10^5$$

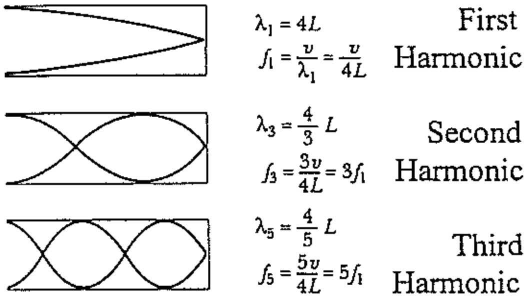
Thus $$\frac{I_2}{I_1} = \frac{10^5}{10^9} = 10^{-4} = \frac{r_1^2}{r_2^2}, \qquad \text{so } r_2 = 10^2 r_1 = 500\,\text{m}$$

16.6 STANDING SOUND WAVES

Standing sound waves can be set up whenever sound is reflected back and forth in an enclosure. In particular, standing sound waves are set up in a column of air, such as in an organ pipe or in a horn. Longitudinal pressure waves are reflected back when they hit an obstruction (for example, the closed end of a pipe) or when they encounter any change in the nature of the structure in which they are propagating. Thus when a sound wave in a pipe encounters the open end of the pipe, it is reflected back. At the closed end of a pipe, the molecules cannot move longitudinally, so this point is a node for displacement (zero displacement). Conversely, a closed pipe end is a point where the pressure variations are large (an antinode). The open end of a pipe is an antinode for displacement and a node for pressure variations. The latter is plausible since the open end is in contact with atmospheric pressure, and this is constant. The above ideas are only approx-imately true, and some corrections have to be made for very accurate calculations. Also, it is assumed that the pipe diameter is small compared to its length. In Figure 16.9 the pressure variation for several modes is shown for a pipe open at both ends and for a pipe closed at one end. The



Pipe open at both ends



Pipe open at one end

frequencies (resonances) are related to the pipe length. For a pipe open at both ends the resonant frequencies are integer multiples of the first harmonic (the fundamental frequency), just as for a string fixed at both ends.

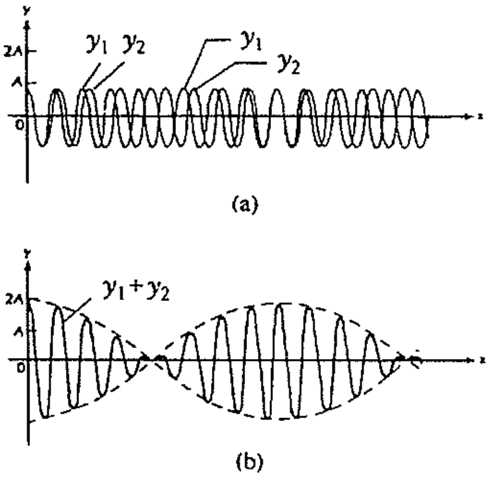
Pipe open at both ends: $$\boxed{f_n = n\frac{v}{2L} \quad (n = 1, 2, 3, \cdots)} \qquad (16.15)$$

In a pipe closed at one end, only odd harmonics are present.

Pipe closed at one end: $$\boxed{f_n = n\frac{v}{4L} \quad (n = 1, 3, 5, \cdots)} \qquad (16.16)$$

16.7 BEATS

We have seen that two traveling waves of the same frequency and velocity can interfere. They can reinforce each other, or, if they have equal amplitudes and are 180° out of phase, they can cancel each other. If they have slightly different frequencies, they interfere and produce a phenomenon called **beats**. In Figure 16.10*a* two waves of slightly different frequency are shown, and their superposition $y = y_1 + y_2$ is graphed in Figure 16.10*b*. Suppose you are at a fixed point in space and the sound wave train shown passes by you. For simplicity, suppose ou are at the origin, and the time dependence of the two superimposed waves is $y_1 = A\cos 2\pi f_1 t$ and $y_2 = A\cos 2\pi f_2 t$. The resultant wave is $y = y_1 + y_2$. Use the trigonometric identity



(a)

(b)

Figure 16.10

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

where $\alpha = 2\pi f_1 t$ and $\beta = 2\pi f_2 t$. Thus

$$y = 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad (16.17)$$

An observer hearing this sound will detect a predominant frequency equal to the average frequency $(f_1 + f_2)/2$ and an amplitude that varies in time at frequency $(f_1 - f_2)/2$, as seen in Figure 16.10*b*. The amplitude is a maximum whenever $\cos 2\pi[(f_1 - f_2)/2] = \pm 1$. Thus there are two maxima per cycle, so one hears **beats** at a beat frequency of $f = f_2 - f_1$. For example, if one tuning fork produces sound at 260 Hz and a nearby one produces 264 Hz, you would hear something that sounded like 262

Hz, but it would get loud and quiet at four times per second. Long ago piano tuners would tune a piano by striking a tuning fork that resonated at a desired frequency (say, low C, 65 Hz) and listen to it while striking the same note on the keyboard. If the piano is out of tune, a beat sound will result. The piano tuner then adjusts the tension of the piano string (and thereby the wave velocity on the string and the fundamental frequency) until the beat sound goes away. Nowadays electronic signal generators are usually used in place of tuning forks.

Note that in a stringed instrument, the sound you hear is not a direct result of the vibrating string. The sound produced by an isolated string is rather weak, as you may have noticed when you hear a bow string released. However, in an instrument like a guitar, the vibrating string causes a sounding board to vibrate, and it is this sounding board that produces the pressure waves we hear as sound.

16.8 THE DOPPLER EFFECT

Suppose you stand on a street corner while a race car is speeding toward you. The sound you hear is something like this: "Eeeeeeeeeeeeuuuuuuunnnnhhhhh." As the car approaches, you hear a high-pitched sound, and as it passes you and moves away, the pitch drops noticeably. Here is what happens: If the source is stationary, it emits crests of the pressure wave at a rate of f times per second (f is the frequency). The crests are separated by a distance λ, the wavelength. They travel toward you at v, the speed of sound. If now the source moves toward you at speed v_s, the spacing between crests is reduced because the source chases after each crest it emits, moving a distance v_sT before emitting the next crest, where $T = 1/f =$ the period of the sound wave (see Figure 16.11). Thus the spacing of the crests coming toward you is $\lambda' = vT - v_sT = \lambda - v_sT = \lambda - v_s/f$. The waves will pass you with frequency f'.

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda(-v_s/f)}$$

But $\lambda = \frac{v}{f}$, so $f' = (\frac{v}{v - v_s})f$

Thus when the source approaches at speed v_s, a higher frequency is heard. When the source moves away, a lower frequency is heard.

$$\boxed{\text{Source approaching } f' = (\frac{v}{v - v_s})f} \qquad (16.18)$$

$$\boxed{\text{Source receding from } f' = (\frac{v}{v + v_s})f} \qquad (16.19)$$

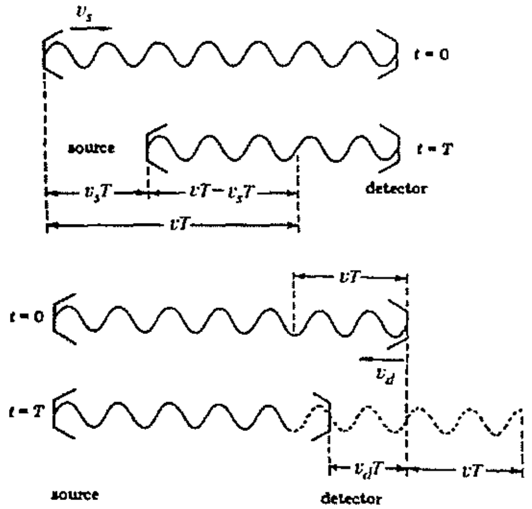


Figure 16.11

If the detector approaches a stationary source, he he will hear a frequency higher than normal (Figure

16.11). Suppose T is the time between crests an observer detects when he and the source are both stationary, where $T = 1/f$ and $v = f\lambda$. If the detector moves with speed v_d toward an oncoming crest, the detector approaches the crest with speed $v + v_d$, so the time to cover the distance from one crest to the next will be shorter; that is, $T' = \lambda/(v + v_d) = 1/f'$. Substitute $\lambda = v/f$ and we obtain $1/f' = v/f(v + v_d)$ or $f' = [(v + v_d)/v]f$. Similar reasoning shows that when the detector moves away from the source, he detects a lower frequency. Thus

$$\boxed{\text{Detector approaching } f' = (\tfrac{v+v_d}{v})f} \tag{16.20}$$

$$\boxed{\text{Detector receding from } f' = (\tfrac{v-v_d}{v})f} \tag{16.21}$$

If both the source and the detector are moving, the frequency detected is

$$\boxed{f' = (\tfrac{v \pm v_d}{v \mp v_s})f} \tag{16.22}$$

Here use the upper sign ($+v_d$ and $-v_s$) if the source and detector are approaching. Use the lower sign ($-v_d$ and $+v_s$) if the source and detector are moving away from each other. f is the frequency when both detector and source are stationary.

Problem 16.8 A stationary police car siren emits a sound at 1200 Hz. Under conditions when the velocity of sound in air is 340 m/s, what frequency will you hear when stationary if the siren is approaching at 30 m/s? What frequency will you hear when the siren is moving away at 30 m/s?

Solution Approaching: $f' = \frac{v}{v - v_s}f = (\frac{340}{340 - 30})(1200\,\text{Hz}) = 1316\,\text{Hz}$

Going away: $f' = \frac{v}{v + v_s}f = (\frac{340}{340 + 30})(1200\,\text{Hz}) = 1103\,\text{Hz}$

16.9 SUMMARY OF KEY EQUATIONS

Wave traveling toward $+x$ axis: $y = A \sin 2\pi\,(\frac{x}{\lambda} - \frac{t}{T})$

or $y = A \sin\,(kx - \omega t)$

Angular frequency: $\omega = 2\pi f = 2\pi\frac{1}{T}$

Wave number: $k = \frac{2\pi}{\lambda}$

For all waves: $v = f\lambda$

Transverse string wave: $v = \sqrt{\frac{T}{\mu}}$ $T = \text{tension}$

$\mu = \frac{\text{mass}}{\text{length}}$

Power transmitted by a string wave: $P = \frac{1}{2}\mu v \omega^2 A^2$

Standing waves in a string fixed at both ends: $f = n\frac{v}{2L}, \quad (n = 1, 2, 3, \cdots)$

Standing waves in a pipe open at both ends: $f = n\frac{v}{2L}, \quad (n = 1, 2, 3, \cdots)$

Standing waves in a pipe open at one end: $f = n\frac{v}{4L}, \quad (n = 1, 3, 5, \cdots)$

Intensity: $I = \frac{\text{power}}{\text{area}}$

Decibel level: $\beta = 10 \log_{10} \frac{I}{I_0}$, where $I_0 = 10^{-12}$ W/m^2

Supplementary Problems

SP 16.1 A transverse string wave travels in the negative x direction with amplitude 0.002 m, frequency 200 Hz, and wavelength 0.20 m. The displacement of the wave is $y = 0$ at $t = 0$ and $x = 0$. Write an expression for the displacement y.

SP 16.2 Humans can hear a range of frequencies from 20 to 20,000 Hz. To what range of wavelengths in air does this correspond (assume the sound velocity is 340 m/s)?

SP 16.3 A copper wire (density 8920 kg/m^3) has diameter 2.4 mm. With what velocity will transverse waves travel along it when it is subjected to a tension of 20 N?

SP 16.4 The G string on a guitar has a length of 0.64 m and a fundamental frequency of 196 Hz. We can effectively shorten the length of the string by pressing it down on a fret (a small ridge on the neck of the guitar). How far should the fret be from the end of the string if you are to produce a fundamental frequency of 262 Hz (the C note)?

SP 16.5 A tuning fork vibrates at 462 Hz. An untuned violin string vibrates at 457 Hz. How much time elapses between successive beats?

SP 16.6 Energy is transmitted at a power level P_1 with frequency f_1 on a string under tension T_1. (a) What will be the power transmitted if the tension is increased to $2T_1$? (b) What will be the power transmitted if the tension remains T_1 but the frequency is increased to $2f_1$?

SP 16.7 A long pipe is closed at one end and open at the other. If the fundamental frequency of this pipe for sound waves is 240 Hz, what is the length of the pipe? Assume sound velocity is 340 m/s in air.

SP 16.8 Suppose a source of sound radiates uniformly in all directions. By how many decibels does the sound level decrease when the distance from the source is doubled?

SP 16.9 What is the intensity level in decibels of a sound whose intensity is 4.0×10^{-7} W/m^2? What is the pressure amplitude of such a wave? Assume sound velocity is 340 m/s in air.

SP 16.10 In my town a siren is sounded at the fire station to call volunteer firemen to duty. If the frequency of the siren is 300 Hz, what frequency would you hear when driving toward the siren at 20 m/s?

Solutions to Supplementary Problems

SP 16.1

$$y = 0.002 \sin 2\pi \left(\frac{x}{0.20} + 200\,t\right)$$

SP 16.2

$$\lambda_1 = \frac{v}{f_1} = \frac{340\,\text{m/s}}{20/\text{s}} = 17\,\text{m}, \qquad \lambda_2 = \frac{340\,\text{m/s}}{20{,}000/\text{s}} = 0.017\text{m}$$

SP 16.3 Consider 1 m of wire: $\mu = \frac{m}{L} = \frac{\rho v}{L} = \rho\frac{AL}{L} = \rho A$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{20\,\text{N}}{(8920\text{kg/m}^3)(\pi)(0.0012\text{m})^2}} = 22.3\text{m/s}$$

SP 16.4

$$f = n\frac{v}{2L}, \qquad n = 1, \qquad \text{so } f_G = \frac{v}{2L_G}, \qquad f_c = \frac{v}{2L_c}$$

$$\frac{f_c}{f_G} = \frac{v/2L_c}{v/2L_G} = \frac{L_G}{L_c}, \qquad L_c = \frac{f_G}{f_c}L_G = \frac{196}{262}(0.64\text{m})$$

$$L_c = 0.48\,\text{m}, \qquad \text{so } \Delta L = L_G - L_c = 0.64\text{ m} - 0.48\text{ m} = 0.16\text{ m}$$

SP 16.5 Beat frequency: $f_B = f_2 - f_1 = 462\text{ Hz} - 457\text{ Hz} = 5\text{ Hz}$

$$T_B = \frac{1}{f_B} = \frac{1}{5}\text{s} = 0.2\,\text{s}$$

SP 16.6

$$P = \tfrac{1}{2}\mu\omega^2 v, \qquad v = \sqrt{\frac{I}{\mu}}, \qquad \text{so } P = \tfrac{1}{2}\mu\omega^2\sqrt{\frac{T}{\mu}} = \tfrac{1}{2}\omega^2\sqrt{\mu T}$$

(a)

$$\frac{P_2}{P_1} = \frac{1/2\,\omega^2\sqrt{\mu(2T_1)}}{1/2\,\omega^2\sqrt{\mu T_1}} = \sqrt{2}, \qquad P_2 = \sqrt{2}\,P_1 = 1.4\,P_1$$

(b)

$$\frac{P_3}{P_1} = \frac{1/2\,\omega_3^2\sqrt{\mu T_1}}{1/2\,\omega_1^2\sqrt{\mu T_1}} = \frac{f_3^2}{f_1^2} = \frac{(2f_1)^2}{f_1^2} = 4, \qquad P_3 = 4P_1$$

SP 16.7

$$f_n = n\frac{v}{4L}, \qquad n = 1, \qquad \text{so } L = \frac{v}{4f_1} = \frac{340\,\text{m/s}}{(4)(240/\text{s})} = 0.35\text{ m}$$

SP 16.8

$$\text{Intensity} = \frac{\text{power}}{\text{area}} = \frac{P}{4\pi r^2}, \qquad \text{so } \frac{I_1}{I_2} = \frac{P/4\pi r_1^2}{P/4\pi r_2^2} = \left(\frac{r_2}{r_1}\right)^2$$

$$\beta_1 = 10\log\frac{I_1}{I_0}, \qquad \beta_2 = 10\log\frac{I_2}{I_0}$$

$$\beta_1 - \beta_2 = 10\log\frac{I_1}{I_0} - 10\log\frac{I_2}{I_0} = 10\log I_1 - 10\log I_0 - 10\log I_2 + 10\log I_0$$

$$= 10\log I_1 - 10\log I_2 = 10\log\frac{I_1}{I_2} = 10\log\left(\frac{r_2}{r_1}\right)^2 = 20\log\frac{r_2}{r_1}$$

If $r_2 = 2r_1$, then $\beta_1 - \beta_2 = 20\log 2 = 6.02\text{ dB}$

SP 16.9

$$\beta = 10\log\frac{I_1}{I_0} = 10\log\frac{4\times 10^{-7}\text{ W/m}^2}{10^{-12}\text{ W/m}^2} = 44\text{ dB}$$

SP 16.10

$$f' = f\left(1 + \frac{v_d}{v}\right) = (300\text{ Hz})\left(1 + \frac{20\text{ m/s}}{340\text{ m/s}}\right) = 318\text{ Hz}$$

CHEMISTRY

REVIEW

REVIEW OF CHEMISTRY

FOR

ENGINEERING ENTRANCE EXAM

Prepared by:

Dr. Sujata Guha

Department of Chemistry

Tennessee State University

Math Concepts in Chemistry:

Algebra

Algebra is often used in chemistry problems to simplify mathematical expressions. The key rule is: Anything you do to one side of an expression, you must also do to the other side of the expression.

Example A: Determine the value of x in the equation $2x+3=9$.

Solution: First, simply subtract 3 from both sides of the equation.

$$2x+3-3=9-3$$

$$2x=6$$

$$\mathrm{x}=6/2 \quad \boldsymbol{x=3}$$

Example B: Solve for x in the equation $\frac{1}{4}x+3+2x=5$.

Solution: Multiply the equation by 4.

$$(4)(\frac{1}{4}x)+(4)(3)+(4)(2x)=(4)(5)$$

Then, simplify $x+12+8x=20$

Combine like terms $x+8x+12-12=20-12$

$$9x=8 \quad \boldsymbol{x=8/9}$$

Example C: Rearrange the equation $\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}$ to find an expression for T_2.

Solution: First flip both sides of the equation in order to have T_2 in the numerator.

$$\frac{T_1}{P_1V_1}=\frac{T_2}{P_2V_2}$$

Then, multiply both sides of the equation by P_2V_2.

$$\frac{T_1}{P_1V_1}(P_2V_2) = \frac{T_2}{P_2V_2}(P_2V_2)$$

Then, $\frac{T_1}{P_1V_1}(P_2V_2) = T_2$

$$\frac{T_1P_2V_2}{P_1V_1} = T_2$$

$$\mathbf{T_2 = T_1P_2V_2/P_1V_1}$$

Example D: Evaluate the expression $3y+1$ when $y = 2$.

Solution: $3y + 1 = 3(2) + 1 = 7$

Exponents

Exponents are a short-hand notation for expressing very large or very small numbers. The exponent gives the number of times that a number should be multiplied by itself. For example, 10^3 indicates that 10 should be multiplied three times by itself:

$$10^3 = (10)(10)(10) = 1000$$

When working with the number 10, changing the exponent by one is the same as moving the decimal point one place.

$$1.000 \text{ x } 10^3 = 10.00 \text{ x } 10^2 = 100.0 \text{ x } 10^1 = 1000$$

$$2.3 \text{ x } 10^{-2} = 0.23 \text{ x } 10^{-1} = 0.023$$

A negative exponent indicates that the result is the inverse of the number raised to the exponent.

$$8^{-2} = \frac{1}{8^2} = \frac{1}{64} = 0.0156$$

When multiplying or dividing numbers with exponents, the exponents are added or subtracted.

$$(10^8)(10^4) = 10^{(8+4)} = 10^{12}$$

$$\frac{10^9}{10^3} = 10^{(9-3)} = 10^6$$

Raising a number with an exponent to another power is the same as same as multiplying the two exponents together.

$$(10^3)^3 = 10^{(3\times3)} = 10^9$$

Square roots are written as $x^{1/2}$ while cube roots as $x^{1/3}$. The square root of 10^6 is:

$$\left(10^6\right)^{\frac{1}{2}} = 10^{\left(6x\frac{1}{2}\right)} = 10^{\frac{6}{2}} = 10^3$$

When adding and subtracting numbers with exponents, first express the numbers with the same exponent, and then perform the operation.

$$(4.00 \text{ x } 10^4 \text{ g}) - (5.00 \text{ x } 10^3 \text{ g})$$

$$= (40.0 \text{ x } 10^3 \text{ g}) - (5.0 \text{ x } 10^3 \text{ g})$$

$$= 35.0 \text{ x } 10^3 \text{ g} = 35{,}000 \text{ g}$$

Example A: Express the number 2,454,000 in exponential notation.

Solution: $2{,}454{,}000 = 2.454 \text{ x } 10^6$

Example B: Express the number 0.0000002623 in exponential notation.

Solution: $0.0000002623 = 2.623 \text{ x } 10^{-7}$

Example C: Perform the addition $(1.1 \text{ x } 10^4) + (2.1 \text{ x } 10^4)$.

Solution: $(1.1 \text{ x } 10^4) + (2.1 \text{ x } 10^4) = (1.1 + 2.1) \text{ x } 10^4 = 3.2 \text{ x } 10^4$

Logarithms

Logarithms are a way of counting in multiples of a base number. If no base is specified, it is assumed to be 10 and is abbreviated as $\log_{10}$ or log.

$\log(100) = \log(10^3) = 3$ $\log(1) = \log(10^0) = 0$ $\log(0.01) = \log(10^{-2}) = -2$

A natural log (ln) uses the base of "e" (e = 2.71828).

$\ln(100) = 4.605$ $\ln(10) = 2.303$ $\ln(0.1) = -2.303$

If an unknown x is associated with log, to determine x, take the antilog of the number on the other side of the equation.

$$\text{if } \log x = 1,$$

$$x = \text{antilog } 1 = \mathbf{10}$$

Example A: Find the value of x, if $\log x = 3$.

Solution: $\log x = 3$

$$x = \text{antilog } 3 = \mathbf{1 \text{ x } 10^3}$$

Example B: Determine the value of p, if $p = -\log(1 \text{ x } 10^{-7})$.

Solution: $p = -\log(1 \text{ x } 10^{-7})$

$$p = -(-7) = \mathbf{7}$$

Significant Figures

The basic rules for counting significant figures are:

(1) All non-zero digits are significant. 1234 has 4 significant figures.

(2) Zeros between non-zero digits are significant. 13.201 has significant figures

(3) Zeros to the left of the first non-zero digit are not significant. 3.02 has 3 significant figures, while 0.11 has 2 significant figures.

(4) If a number ends in zeros to the right of the decimal point, those zeros are significant. 1.0 has 2 significant figures, while 3.00 has 3 significant figures.

(5) If a number ends in zeros to the left of the decimal point, those zeros may or may not be significant.

(6) For multiplication or division, the number of significant figures in the answer is the same as the number of significant figures in the least precise measurement or limiting term. $(3.21)(1.1) = 3.531 = 3.5$. The final answer is expressed in 2 significant figures since the limiting term, 1.1, has 2 significant figures.

(7) For addition or subtraction, the answer should have the same number of decimal places as the number of significant figures in the least precise measurement. 10.1 + 21.0 + 1.021 = 32.121 = 32.1. The final answer is expressed with one decimal place since the limiting term, 21.0, has 1 decimal place.

Example A: Determine the number of significant figures in 0.00023.

Solution: The answer has **2 significant figures.**

Example B: Determine the number of significant figures in the operation $4.220 \times 10^{-6} - 9.963 \times 10^{-7}$.

Solution: First, convert both numbers to the same exponent

$$4.220 \times 10^{-6} - 9.963 \times 10^{-7} = 4.220 \times 10^{-6} - 0.9963 \times 10^{-6}$$
$$= 3.224 \times 10^{-6} = 0.000003224$$

The answer has **4 significant figures**.

Dimensional Analysis

The following is a list of basic conversions:

1 in = 2.54 cm

1 km = 1000 m

1 m = 1.094 yd

1 mi = 1760 yd

1 kg = 2.205 lb

1 nm = 1×10^{-9} m

1 m = 100 cm

1 m = 1000 mm

1 m = 10 dm

1 mi = 5280 ft

12 in = 1 ft

1 ft^3 = 28.32 L

1 h = 60 min

1 d = 24 h

1 L = 1000 mL

1 min = 60 s

1 μg = 1 x 10^{-6} g

1 qt = 32 oz

1 kg = 1000 g

1 gal = 4 qt

1 lb = 453.6 g

1 L = 1.06 qt

Example A: Convert 2 feet to inches.

Solution: (2 ft)(12 in/1 ft) = **24 in**

Example B: Calculate the number of ounces in 3 gallons?

Solution: (3 gal)(4 qt/1 gal)(32 oz/1 qt) = **384 oz**

Example C: Convert 3 cubic feet to cubic centimeters.

Solution: $(3\ \mathrm{ft}^3)(12\ \mathrm{in}/1\ \mathrm{ft})^3(2.54\ \mathrm{cm}/1\ \mathrm{in})^3$

$= [(3\ \mathrm{ft}^3)(12\ \mathrm{in})^3(2.54\ \mathrm{cm})^3]/[(1\ \mathrm{ft})^3(1\ \mathrm{in})^3]$

$= 84913.92\ \mathrm{cm}^3 = \mathbf{8.4 \times 10^4\ cm^3}$

Practice Problems:

Practice problem A: Express 0.00000452 and 332,000 in exponential notation.

Practice problem B: Evaluate $(3^5)(3^3)$.

Practice problem C: Evaluate $\log(1.2 \times 10^6)^3$.

Practice problem D: Evaluate $\ln(1.3 \times 10^{-3})$.

Practice problem E: If $\log y = -10.4$, determine y.

Practice problem F: Evaluate $\frac{12}{6/4}$.

Practice problem G: How many significant figures does the number 0.509 have?

Practice problem H: Express the product (2.0)(10.50) to the correct number of significant figures.

Practice problem I: Convert 2.5 nm to m.

Practice problem J: Convert 25 mi/h to ft/min.

Density:

Density of a substance is defined as a ration of the mass of the substance to its volume.

Density = Mass/Volume

This equation can be rearranged to calculate volume and mass

Volume = Mass/Density

Mass = (Volume)(Density)

Example A: Calculate the density of 28 mL of ethanol, whose mass is 26.4 g.

Solution: mass = 26.4 g and volume = 28 mL

Thus, density = 26.4 g/28mL = **0.94 g/mL**

Example B: A substance has a density of 0.96 g/cm^3 at room temperature. Calculate the mass of 5 cm^3 of the substance.

Solution: density = 0.96 g/cm^3 and volume = 5 cm^3

Thus, mass = (0.96 g/cm^3)(5 cm^3) = **4.8 g**

Example C: Graphite has a density of 3.5 g/cm^3. Calculate the volume of 0.5 g of it.

Solution: density = 3.5 g/cm^3 and mass = 0.5 g

Thus, volume = (0.5 g)/(3.5 g/cm^3) = **0.14 cm^3**

Example D: A 15 g piece of plastic weighing is immersed in 10 mL methanol in a flask. The final volume of methanol with the plastic in it is 16.0 mL. Calculate the density of the plastic.

Solution: Mass of plastic = 15 g

Volume of methanol = 10 mL

Volume of methanol + plastic = 16 mL

So, Volume of plastic = 16 mL - 10 mL = 6 mL

Density = mass/volume = 15 g/6 mL = **2.5 g/mL**

Practice Problems:

Practice problem A: Calculate the density of a sample of emerald which has a volume of 1.6 cm^3 and a mass of 6.7 g.

Practice problem B: Calculate the density of a liquid which has a volume of 28 mL and a mass of 26.4g.

Practice problem C: Milk has a density of 1.03 g/mL. Calculate the mass of 1 L of milk.

Practice problem D: If the mass of barium is 65.4 g and its volume is 32.1 cm^3, what would be its density?

Practice problem E: Calculate the mass of 30.0 mL of methanol, given the density of methanol is 0.790 g/mL.

Practice problem F: A 200 cm^3 of salt weighs 433 grams. What is its density?

Practice problem G: A piece of unknown material has a mass of 5.854 g and a volume of 7.57 cm^3. What is the density of the material?

Practice problem H: Iron has a known density of 7.87 g/cm^3. What would be the mass of a 2.5 dm^3 sample of iron?

Practice problem I: Mercury has a density of 13.5 g/cm^3. How much volume would 50.0 g of mercury occupy?

Practice problem J: Pure gold has a density of 19.32 g/cm^3. Calculate the volume of 318.97 g of it.

Interconversion between Mass and Moles:

To calculate the number of moles of a substance, use the equation

# of moles = mass/molar mass

This equation can be rearranged to calculate the mass of a substance, given the # of moles.

mass = (# of moles)(molar mass)

Example A: Determine the number of moles of CO_2 in 454 g.

Solution: Molar mass of CO_2 = 12.01 g/mol + 2(16.00 g/mol) = 44.01 g/mol

# of moles n = 454 g/44.01 ~~g~~/mol = **10.3 mol**

Example B: How many moles of sulfur dioxide, SO_2, are in 2000 grams of the gas?

Solution: Molar mass of SO_2 = 32.06 g/mol + 2(16.00 g/mol) = 64.06 g/mol

# of moles n = 2000 g/64.06 ~~g~~/mol = **31.2 mol**

Example C: How many grams of hydrogen (H_2) are needed to give 3 moles of it?

Solution: Molar mass of H_2 = 2(1.008 g/mol) = 2.016 g/mol

mass = (3 mol)(1.016 g/mol) = **6.048 g**

Example D: Determine the mass of H_2SO_4 in 3.60 moles.

Solution:

Molar mass of H_2SO_4 = 2(1.008 g/mol) + 32.06 g/mol + 4(16.00 g/mol) = 98.08 g/mol

mass = (3.60 mol)(98.08 g/mol) = **353 g**

Practice Problems:

Practice problem A: How many moles are present in 41.2 g of $Mg_3(PO_4)_2$?

Practice problem B: Calculate the mass of 5.10 moles of sulfur (S).

Practice problem C: How many moles are present in 26.9 g of $C_{12}H_{22}O_{11}$?

Practice problem D: A certain laboratory uses 0.100 moles of magnesium (Mg). How many grams of magnesium would you weigh out?

Practice problem E: How many moles of KBr are present in 14.0 grams?

Practice problem F: How many grams of CO_2 are found in 1.50 moles?

Practice problem G: If you find the mass of a sample of glucose ($C_6H_{12}O_6$) to be 50.0 g, how many moles of glucose do you have?

Practice problem H: Convert 3.57 moles of aluminum (Al) to grams.

Practice problem I: What is the mass of 4.26 moles of silicon (Si)?

Practice problem J: How many moles are present in 17.7 g of $KMnO_4$?

Percent by Mass:

% by mass = (mass of substance in compound/total mass of compound) x 100

If, instead of a single compound you have a mixture of compounds, then

% by mass = (mass of substance/total mass of solution or mixture) x 100

Percent by mass is sometimes referred to as percent by weight.

Example A: Calculate the percent by mass of sodium and chlorine in sodium chloride (NaCl)

Solution: Mass of Na (from Periodic Table) = 22.99 g/mol

Mass of Cl (from Periodic Table) = 35.45 g/mol

Molar mass of NaCl = 22.99 g/mol + 35.45 g/mol = 58.44 g/mol

%Na = (22.99 g/mol/58.44 g/mol) x 100 = **39.34%**

%Cl = (35.45 g/mol/58.44 g/mol) x 100 = **60.66%**

Example B: Calculate the percent by weight of each element present in sodium sulfate (Na_2SO_4).

Solution: Mass of Na (from Periodic Table) = 22.99 g/mol

Mass of S (from Periodic Table) = 32.06 g/mol

Mass of O (from Periodic Table) = 16.00 g/mol

Molar mass of Na_2SO_4 = 2(22.99 g/mol) + 32.06 g/mol + 4(16.00 g/mol) = 142.04 g/mol

%Na = [2(22.99 g/mol)/142.04 g/mol] x 100 = **32.37%**

%S = (32.06 g/mol/142.04 g/mol) x 100 = **22.57%**

%O = [4(16.00 g/mol/142.04 g/mol)] x 100 = **45.06%**

Example C: Calculate the percent by weight of each element present in ammonium phosphate [$(NH_4)_3PO_4$]

Solution: Mass of N (from Periodic Table) = 14.01 g/mol

Mass of H (from Periodic Table) = 1.008 g/mol

Mass of P (from Periodic Table) = 30.97 g/mol

Mass of O (from Periodic Table) = 16.00 g/mol

Molar mass of $(NH_4)_3PO_4$ = 3(14.01 g/mol) + 12(1.008 g/mol) + 30.97 g/mol + 4(16.00 g/mol) = 149.096 g/mol

%N = [3(14.01 g/mol)/149.096 g/mol] x 100 = **28.19%**

%H = [12(1.008 g/mol/149.06 g/mol)] x 100 = **8.11%**

%P = (30.97 g/mol/149.06 g/mol) x 100 = **20.77%**

%O = [4(16.00 g/mol/149.06 g/mol)] x 100 = **42.93%**

Example D: Dry air contains roughly 78.09% nitrogen, 20.95% oxygen, 0.93% argon, 0.039% carbon dioxide, and small amounts of other gases. Calculate the mass percent of oxygen.

Solution: Since percentages are given, assume that the total mass of air is 100 g so that the mass of each element = the percent given.

mass of oxygen = 20.95 g (given).

%O = (20.95 g/100 g) x 100 = **20.95%**

Practice Problems:

Practice problem A: What is the weight percent of glucose in a solution made by dissolving 5.3 g of glucose in 138.2 g of water?

Practice problem B: What is the weight percent of methanol in a solution prepared by dissolving 2.0 g of methanol in 90.0 g of water?

Practice problem C: A sample of a solution weighing 700.0 g is known to contain .223 moles of potassium chloride (KCl). What is the percent by weight of potassium chloride in the solution?

Practice problem D: Bicarbonate of soda, $NaHCO_3$, is used in many commercial preparations. Find the mass percentages of Na, H, C, and O in $NaHCO_3$.

Practice problem E: If 67.1 g of $CaCl_2$ are added to 275 g of water, calculate the mass percent of $CaCl_2$ in the solution.

Practice problem F: A 5 g sugar cube ($C_{12}H_{22}O_{11}$) is dissolved in a 400 mL container of water. Calculate the percent composition by mass of the sugar solution. Density of water = 1.0 g/mL.

Practice problem G: Calculate the mass% of 50 g of NaCl in 180 g of solution?

Practice problem H: Determine the percent composition by mass of a 100 g salt solution which contains 15 g salt.

Practice problem I: What are the mass percent of carbon and oxygen in carbon monoxide, CO?

Practice problem J: Calculate the mass percent composition of nitrogen in N_2O_3.

Empirical Formula:

This is a formula that gives the simplest whole-number ratio of atoms in a compound. The following steps should be employed to determine empirical formula.

(1) Start with the number of grams of each element, given in the problem. If percentages are given, assume that the total mass is 100 grams so that the mass of each element = the percent given.

(2) Convert the mass of each element to moles using the molar mass from the periodic table.

(3) Divide each mole value by the smallest number of moles calculated and round to the nearest whole number. This is the mole ratio of the elements and is represented by subscripts in the empirical formula. If the number is too far to round, multiply each solution by the same factor to get the lowest whole number multiple. e.g. If one solution is 1.5, then multiply each solution in the problem by 2 to get 3, if one solution is 1.25, then multiply each solution in the problem by 4 to get 5, if one solution is 2.5, then multiply each solution in the problem by 2 to get 5.

Example A: A compound was analyzed and found to contain 13.5 g Ca, 10.8 g O, and 0.675 g H. What is the empirical formula of the compound?

Solution: # mol of Ca = (13.5 g)(1 mol Ca/40.1 g) = 0.337 mol Ca

mol of O = (10.8 g)(1 mol Ca/16.0 g) = 0.675 mol O

mol of H = (0.675 g)(1 mol Ca/1.01 g) = 0.668 mol H

The smallest # of moles is 0.337. Divide each mole value by this number. Round answer to the nearest whole number.

0.337 mol Ca/0.337 = 1

0.675 mol O/0.337 = 2

0.668 mol H/0.337 = 1.98 ≈ 2

This is the mole ratio of the elements and is represented by subscripts in the empirical formula. Thus, the empirical formula is $Ca_1O_2H_2$ or **$Ca(OH)_2$**.

Example B: NutraSweet is 57.14% C, 6.16% H, 9.52% N, and 27.18% O. Calculate the empirical formula of NutraSweet and find the molecular formula. (The molar mass of NutraSweet is 294.30 g/mol)

Solution: Since percentages are given, assume that the total mass is 100 grams so that the mass of each element = the percent given.

Thus, we have 57.14 g C, 6.16 g H, 9.52 g, N, and 27.18 g O.

mol of C = (57.14 g)(1 mol C/12.0 g) = 4.76 mol C

mol of H = (6.16 g)(1 mol H/1.01 g) = 6.10 mol H

mol of N = (9.52 g)(1 mol N/14.0 g) = 0.68 mol N

mol of O = (27.18 g)(1 mol O/16.0 g) = 1.70 mol O

The smallest # of moles is 0.68. Divide each mole value by this number. Round answer to the nearest whole number.

4.76 mol C/0.68 = 7

6.10 mol H/0.68 = 8.97 ≈ 9

0.68 mol N/0.68 = 1

1.70 mol O/0.68 = 2.5 = 5/2

Multiply each solution by the factor 2 to get the lowest whole number multiple.

7(2) = 14

9(2) = 18

1(2) = 2

5/2(2) = 5

Thus, the empirical formula is $\mathbf{C_{14}H_{18}N_2O_5}$.

Example C: The composition of a compound is 40% sulfur and 60% oxygen by weight. What is its empirical formula?

Solution: Since percentages are given, assume that the total mass is 100 grams so that the mass of each element = the percent given.

Thus, we have 40.0 g S and 60.0 g O.

mol of S = (40.0 g)(1 mol C/32.1 g) = 1.25 mol S

mol of O = (60.0 g)(1 mol O/16.0 g) = 3.75 mol O

The smallest # of moles is 1.25. Divide each mole value by this number. Round the answer to the nearest whole number.

1.25 mol S/1.25 = 1

3.75 mol O/1.25 = 3

Thus, the empirical formula is S_1O_3 or **SO_3**.

Practice Problems:

Practice problem A: Determine the empirical formula of methane that is composed of 4.5 g of carbon (C) and 1.5 g of hydrogen (H).

Practice problem B: Determine the empirical formula of the compound made when 8.65 g of iron (Fe) combines with 3.72 g of oxygen (O).

Practice problem C: Pure formaldehyde consists of 40.0% carbon, 6.7% hydrogen, and 53.3% oxygen. What is its empirical formula?

Practice problem D: Determine the empirical formula of a compound that is 29.0% sodium (Na), 40.5% sulfur (S), and 30.4 % oxygen (O) by weight.

Practice problem E: In an analysis of boron oxide, it was found that 10.0g of boron oxide contained 3.14g of boron (B). The remaining was oxygen (O). Calculate the empirical formula of boron oxide.

Practice problem F: Find the empirical formula of magnesium oxide, given 0.074 g of Mg and 0.046 g of O.

Practice problem G: Determine the empirical formula of a compound containing 0.9 g of calcium (Ca) and 1.6 g of chlorine (Cl).

Practice problem H: The chemical composition of tanzanite is 14.04 % calcium (Ca), 14.17 % aluminum (Al), 14.75 % silicon (Si), 54.59 % oxygen (O) and 2.45 % hydrogen (H). Calculate its empirical formula.

Practice problem I: Phenol is a carbon-hydrogen-oxygen compound, composed of 76.69% carbon, 6.38% hydrogen, and 16.93% oxygen. Determine its empirical formula.

Practice problem J: A hydrocarbon contains 83.7% carbon and 16.3% hydrogen by mass. Determine the empirical formula of the compound.

Types of Chemical Reactions:

There are 5 common types of chemical reactions, as follows:

1. Decomposition reactions
2. Combination reactions
3. Single-replacement reactions
4. Double-replacement reactions
5. Combustion reactions

Decomposition Reactions

A decomposition reaction is one in which a single compound decomposes to two or more other substances. For example the industrial preparation of lime (calcium oxide) involves the decomposition of calcium carbonate by heating it.

$$CaCO_3(s) \rightarrow CaO(s) + CO_2(g)$$

Combination Reactions

A combination reaction is one in which two substances combine to form a third. The reaction of calcium oxide with sulfur dioxide to form calcium sulfite is an example of a combination reaction.

$$CaO(s) + SO_2(g) \rightarrow CaSO_3(s)$$

Single-Replacement Reactions

A **single-replacement** reaction is one in which an element reacts with a compound and replaces another element in the compound. The reaction in which copper displaces silver from an aqueous solution of silver nitrate is an example of a single-replacement reaction.

$$Cu(s) + 2AgNO_3(aq) \rightarrow Cu(NO_3)_2(aq) + 2Ag(s)$$

Double-Replacement Reactions

A double-replacement reaction is one in which there is an exchange of positive ions between two compounds. These reactions generally take place between two ionic compounds in aqueous solution. Precipitation reactions are one type of double-replacement reaction. An example is

$$AgNO_3(aq) + NaCl(aq) \rightarrow AgCl(s) + NaNO_3(aq)$$

Neutralization reactions are another type of double replacement reactions. Such reactions occurs between an acid and a base with the formation of an ionic compound and water. An example is

$$HCl(aq) + NaOH(aq) \rightarrow H_2O(l) + NaCl(aq)$$

Combustion Reactions

A combustion reaction is one in which a substance reacts with oxygen, usually with the rapid release of heat. Organic compounds usually burn in the oxygen in air to produce carbon dioxide and water. For example butane burns in air as follows.

$$2C_4H_{10}(g) + 13O_2(g) \rightarrow 8CO_2(g) + 10H_2O(l)$$

Practice Problems:

Classify each of the following reactions:

A: $2HgO \rightarrow 2Hg + O_2$

B: $4Li + O_2 \rightarrow 2Li_2O$

C: $2N_2O_5 \rightarrow O_2 + 4NO_2$

D: $CaO + H_2O \rightarrow Ca(OH)_2$

E: $CH_3OH + O_2 \rightarrow CO_2 + H_2O$

F: $AgNO_3 + HCl \rightarrow AgCl + HNO_3$

G: $CH_4 + O_2 \rightarrow CO_2 + H_2O$

H: $Fe_2O_3 + HCl \rightarrow FeCl_3 + H_2O$

I: $Zn + CuCl_2 \rightarrow ZnCl_2 + Cu$

J: $Cl_2 + NaBr \rightarrow NaCl + Br_2$

Solution Concentration:

A solution is a homogeneous mixture of two or more substances. In a solution, the solvent is the major component (present in greater amount), while the solute is the minor component (present in smaller amount). The strength or concentration of a solution is often expressed in terms of the amount of solute in a specific amount of solvent. The most common unit of concentration is **molarity** (M), defined as the number of moles of solute per unit volume of solution.

Molarity = # of moles of solute / liters of solution

The process of making a solution less concentrated by adding a solvent is called dilution. Calculations involving dilution employ the primary equation

$$M_1V_1 = M_2V_2$$

where M_1 = initial concentration of a substance
V_1 = initial volume of the substance
M_2 = final concentration of the substance
V_2 = final volume of the substance

Example A: Calculate is the molarity of a solution containing 0.32 mole of NaCl in 3.4 liters.

Solution: Molarity = 0.32 mol / 3.4 L = 0.094 mol/L = 0.094 M

Thus, the molarity of the solution is **0.094 M**.

Example B: A 4 g molecule ($C_{12}H_{22}O_{11}$) is dissolved in a 350 mL teacup filled with hot water. What is the molarity of the solution?

Solution:

Molar mass of $C_{12}H_{22}O_{11}$ = 12(12.01 g/mol) + 22(1.01 g/mol) + 11(16.00 g/mol)
= 144.12 g/mol + 22.22 g/mol + 176 g/mol
= 342.34 g/mol

# of moles = mass / molar mass

# of moles of $C_{12}H_{22}O_{11}$ = 4 g/(342.34 g/mol) = 0.01168 mol

Volume of solution = (350 mL)(1 L / 1000 mL) = 0.35 L

Molarity = 0.01168 mol/0.35 L = 0.03 mol/L = 0.03 M

Thus, the molarity of the solution is **0.03 M**.

Example C: How many moles of salt are contained in 300 mL of a 0.40 M NaF solution?

Solution: 1Volume of solution = (300 mL)(1 L / 1000 mL) = 0.3 L

# of moles of NaF = (0.40 M)(0.3 L)
= (0.40 mol/L)(0.3 L)
= 0.12 mol

Thus, there are **0.12 moles** of NaF.

Example D: What volume, in mL, of 18.0 M nitric acid is needed to prepare 2.50 L of a 1.00 M solution?

Solution: $M_1V_1 = M_2V_2$

$M_1 = 1.00$ M
$V_1 = 2.50$ L
$M_2 = 18.0$ M
$V_2 = ?$

$(1.00 \text{ M})(2.50 \text{ L}) = (18.0 \text{ M})(V_2)$

$V_2 = (1.0 \text{ M})(2.50 \text{ L})/(18.0 \text{ M})$
$V_2 = 0.139$ L

$V_2 = (0.139 \text{ L})(1000 \text{ mL}/1 \text{ L})$
$V_2 = 139$ mL

Thus, **139 mL** of nitric acidwill be needed.

Example E: During the course of an experiment, a 0.300 M KCl solution, with an initial volume of 25.0 mL, is diluted to a total final volume of 26.0 mL What is the concentration of KCl in the final solution?

Solution: $M_1V_1 = M_2V_2$

$M_1 = 0.300$ M
$V_1 = 25.0$ mL
$M_2 = ?$
$V_2 = 26.0$ mL

$$(0.300 \text{ M})(25.0 \text{ mL}) = (M_2)(26.0 \text{ mL})$$

$$M_2 = (0.300 \text{ M})(25.0 \text{ mL}) / (26.0 \text{ mL})$$
$$M_2 = 0.288 \text{ M}$$

Thus, the final concentration of KCl is **0.288 M**.

Practice Problems:

Practice problem A: What is the molarity of 3 moles of solute dissolved in 150 mL of solvent?

Practice problem B: How many grams of $NaNO_3$ will be required to prepare 200 mL of a 0.5 M solution?

Practice problem C: A chemist dissolves 92 g of $FeSO_4$ in enough water to make 1.0 L of solution. What is the molarity of the solution?

Practice problem D: How many moles of Na are in 50 of a 1.23 M KBr solution?

Practice problem E: How many grams of Kl are needed to make 25 mL of a 0.1 solution?

Practice problem F: Calculate the volume of a 1.5M solution containing 0.9moles of sodium chloride.

Practice problem G: You have 50mL of a 2M solution of Kl, but 0.9M solution is needed. How many mL of 0.9M can you make?

Practice problem H: A chemist requires 150 of 0.5 KOH solution, but has a 15 M stock solution. What volume of the stock solution will he need to prepare 0.5 NaOH?

Practice problem I: 10 mL of a 1 M solution are diluted to make a 0.4 solution. What is the volume of the resulting solution?

Practice problem J: 200 L of a 4 solution are diluted to 1L. What is the concentration of the resulting solution?

Oxidation Number:

The oxidation number of an element indicates the number of electrons lost or gained due to chemical bonding. Oxidation number is also referred to as oxidation state. A change in the oxidation number indicates whether the element has undergone oxidation or reduction. An increase in the oxidation number of an element indicates oxidation. A decrease in oxidation number of an element indicates reduction.

For an ionic compound, the oxidation number of a cation or anion (monatomic or polyatomic) is the charge associated with the cation or anion. The following tables represent the common cations and anions.

Table: Monatomic Ions

Cation	Name	Anion	Name
H^+	Hydrogen	H^-	Hydride
Li^+	Lithium	F^-	Fluoride
Na^+	Sodium	Cl^-	Chloride
K^+	Potassium	Br^-	Bromide
Cs^+	Cesium	I^-	Iodide
Be^{2+}	Beryllium	O^{2-}	Oxide
Mg^{2+}	Magnesium	S^{2-}	Sulfide
Ca^{2+}	Calcium	N^{3-}	Nitride
Ba^{2+}	Barium	P^{3-}	Phosphide
Al^{3+}	Aluminum		
Ag^+	Silver		
Cd^{2+}	Cadmium		
Ga^{3+}	Gallium		

Table: Binary Monatomic Ions

Ion	Name
Fe^{3+}	Iron(III) or Ferric
Fe^{2+}	Iron(II) or Ferrous
Cu^{2+}	Copper(II) or Cupric
Cu^+	Copper(I) or Cuprous
Co^{3+}	Cobalt(III)
Co^{2+}	Cobalt(II)
Sn^{4+}	Tin(IV)
Sn^{2+}	Tin(II)
Pb^{4+}	Lead(IV)
Pb^{2+}	Lead(II)
Hg^{2+}	Mercury(II)
Zn^{2+}	Zinc

Table: Polyatomic Ions

Ion	Name	Ion	Name
Hg_2^{2+}	Mercury(I)	NCS^-	Thiocyanate
NH_4^+	Ammonium	CO_3^{2-}	Carbonate
NO_2^-	Nitrite	HCO_3^-	Hydrogen Carbonate
NO_3^-	Nitrate	ClO^-	Hypochlorite
SO_3^{2-}	Sulfite	ClO_2^-	Chlorite
SO_4^{2-}	Sulfate	ClO_3^-	Chlorate
HSO_4^-	Hydrogen Sulfate	$C_2H_3O_2^-$	Acetate
OH^-	Hydroxide	MnO_4^-	Permanganate
CN^-	Cyanide	$Cr_2O_7^{2-}$	Dichromate
PO_4^{3-}	Phosphate	CrO_4^{2-}	Chromate
HPO_4^{2-}	Hydrogen Phosphate	O_2^{2-}	Peroxide
$H_2PO_4^-$	Dihydrogen Phosphate	$C_2O_4^{2-}$	Oxalate

The following rules are used to assign an oxidation number to an element.

(1) The oxidation number of an atom in the elemental state is zero. For example, the oxidation number of Br (in gaseous Br_2), O (in gaseous O_2 molecule), and Na (in solid Na atom) are 0.

(2) The oxidation number of an atom in a monatomic ion is equal to its charge. For example, the oxidation number of K in K^+ is +1, while that of F in F^- is −1.

(3) In compounds, the group 1 elements have an oxidation number of +1 and the group 2 elements have an oxidation number of +2. For example, the oxidation number of K (group 1 element) is +1, while the oxidation number of Mg (group 2 element) is +2.

(4) The oxidation number of hydrogen in a compound is +1, except when hydrogen forms hydrides with active metals; then it is −1. For example, the oxidation number of H is +1 in H_2O, but −1 in NaH (sodium hydride).

(5) In binary compounds with metals, group 7 elements have an oxidation number of −1, group 6 elements have an oxidation number of −2, and group 5 elements have an oxidation number of −3. For example, the oxidation number of Cl is −1 in both HCl and PCl_3, the oxidation number of S is −2 in H_2S, and the oxidation number of N is +3 in NH_3.

(6) The oxidation number of oxygen in a compound is −2, except in peroxides (then it is −1) and when combined with fluorine (then it is +2). For example, the oxidation number of O in H_2O is −2, but the oxidation number of O in H_2O_2 (hydrogen peroxide) is −1.

(7) The sum of the oxidation numbers in the formula of a neutral compound is zero. For example, the oxidation numbers of the atoms in HF add up to 0, since HF is a neutral compound. Similarly, the sum of the oxidation numbers in the formula for a polyatomic ion is equal to the charge on that ion. For example, the oxidation numbers of the atoms in SO_4^{2-} add up to −2.

Example A: What is the oxidation number of SO_4 in Na_2SO_4?

Solution: SO_4 is in the form of SO_4^{2-} (sulfate ion) in Na_2SO_4 (see table of polyatomic ions). Thus, the **oxidation number of SO_4 is −2.**

Example B: What is the oxidation number of ClO_3 in $KClO_3$?

Solution: ClO_3 is in the form of ClO_3^- (chlorate ion) in $KClO_3$ (see table of polyatomic ions). Thus, the **oxidation number of ClO_3 is −1.**

Example C: Calculate the oxidation number of carbon in CO.

Solution: According to rule 6, the oxidation number of O is −2. Let the oxidation number of C be x. According to rule 7, the sum of the oxidation numbers of all elements in CO should add up to be zero. Take into account the number(s) of atoms of each element while doing the addition.

$$\begin{aligned} 1(x) + 1(-2) &= 0 \\ x + (-2) &= 0 \\ x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Thus, the **oxidation number of C is +2.**

Example D: What is the oxidation number of nitrogen in NO_2?

Solution: According to rule 6, the oxidation number of O is −2. Let the oxidation number of N be x. According to rile 7, the sum of the oxidation numbers of all elements in a polyatomic ion should add up to be the charge of the ion. Thus,

$$\begin{aligned} 1(x) + 2(-2) &= 0 \\ x + (-4) &= 0 \end{aligned}$$

$$x - 4 = 0$$
$$x = 4$$

Thus, the **oxidation number of N is +4.**

Practice Problems:

Practice problem A: Calculate the oxidation number of Fe in $FeCl_2$.

Practice problem B: Calculate the oxidation number of C in $C_2O_4^{2-}$.

Practice problem C: Determine the oxidation state of Ba in $Ba(NO_3)_2$.

Practice problem D: Determine the oxidation state of F in NF_3.

Practice problem E: Determine the oxidation state of NH_4 in $(NH_4)_2SO_4$.

Practice problem F: What is the oxidation state of SO_4 in $CuSO_4$?

Practice problem G: What is the oxidation number of Br in Br_2?

Practice problem H: Calculate the oxidation number of Cl in ClO_2^-.

Practice problem I: What is the oxidation number of Sn in Sn^{2+}?

Practice problem J: What is the oxidation number of Ag in AgCl?

Gas Laws:

At the end of the 18th century, scientists began to realize that relationships between the pressure, volume and temperature of a sample of gas could be obtained which would hold for all gases. As the result of many different scientists and experiments, several gas laws have been discovered. These laws relate the various state variables (pressure, volume, and temperature) of a gas. The two most common gas laws are Boyle's Law and Charles' Law.

Boyle's Law describes the inversely proportional relation between pressure and volume of a gas, keeping the temperature constant, i.e. $P \alpha 1/V$. As the pressure on a gas increases, at constant temperature, its volume decreases. Some practical applications are:

- The bubbles exhaled by a scuba diver grow as the approach the surface of the ocean. (The pressure exerted by the weight of the water decreases with depth, so the volume of the bubbles increases as they rise.)
- Deep sea fish die when brought to the surface. (The pressure decreases as the fish are brought to the surface, so the volume of gases in their bodies increases, and pops bladders, cells, and membranes).
- Pushing in the plunger of a plugged-up syringe decreases the volume of air trapped under the plunger.

The mathematical expression of Boyle's Law is

$$P_1V_1 = P_2V_2 \quad \text{(at constant T)}$$

where P_1 is the initial pressure

V_1 is the initial volume

P_2 is the final pressure

V_2 is the final volume

T is the temperature

Example A: A sample of helium gas at 25°C is compressed from 200 cm^3 to 0.240 cm^3. Its pressure is now 3.00 cm Hg. What was the original pressure of the helium?

Solution: $P_1 = ?; V_1 = 200\ cm^3$

$P_2 = 3.00\ cm\ Hg; V_2 = 0.240\ cm^3$

$$P_1V_1 = P_2V_2$$

$P_1 = P_2V_2/V_1$

$P_1 = (3.00\ \text{cm Hg})(0.240\ \text{cm}^3)/200\ \text{cm}^3$

$\mathbf{P_1 = 3.60 \times 10^{-3}\ cm\ Hg}$

Example B: A given mass of a gas occupies 240 mL at 800 mm of Hg. What volume will the gas occupy if the pressure is increased to 1200 mm of Hg, temperature remaining constant?

Solution: $P_1= 800$ mm Hg; $P_2=1200$ mm Hg

$V_1=240$ mL; $V_2=?$

$P_1V_1= P_2V_2$

$V_2 = P_1V_1/P_2$

$V_2 = (800\ \text{mm Hg})(240\ \text{mL})/1200\ \text{mm Hg}$

$\mathbf{V_2 = 160\ mL}$

Example C: At a pressure of 2 atmospheres a fixed mass of hydrogen occupies a volume of 8 litres. What pressure must be maintained if the volume is to be increased to 10 litres, temperature remaining constant?

Solution: $P_1 = 2$ atm; $P_2 =?$

$V_1= 8$ L; $V_2=10$ L

$P_1V_1= P_2V_2$

$P_2 = P_1V_1/V_2$

$P_2 = (2\ \text{atm})(8\ \text{L})/10\ \text{L}$

$\mathbf{P_2 = 1.6\ atm}$

Charles' Law describes the directly proportional relation between volume and temperature of a gas, keeping the pressure constant, i.e. $V \alpha T$. As a gas is heated, at constant pressure, its volume increases. Some practical applications are:

- A football inflated inside and then taken outdoors on a winter day shrinks slightly.
- A slightly underinflated rubber life raft left in bright sunlight swells up (Why shouldn't you overinflate your life raft when your ship goes down in tropical waters?)
- The plunger on a turkey syringe thermometer pops out when the turkey is done (The volume of air trapped under the plunger increases when the temperature inside the turkey climbs).

The mathematical expression of Charles' Law is

$$V_1/T_1 = V_2/T_2 \quad \text{(at constant P)}$$

where V_1 is the initial volume

V_2 is the final volume

T_1 is the initial temperature

T_2 is the final temperature

P is the pressure

Example A: If the temperature of a 3.5 L sample of helium gas is increased from 300 K to 900 K, what is its new volume?

Solution:

$V_1 = 3.5$ L; $T_1 = 300$ K

$V_2 = ?$; $T_2 = 900$ K

$V_1/T_1 = V_2/T_2$

$V_2 = V_1T_2/T_1$

$V_2 = (3.5 \text{ L})(900 \text{ K})/(300 \text{ K})$

$\mathbf{V_2 = 10.5\ L}$

Example B: A given mass of a gas occupies 960 mL at 27°C. What volume will it occupy if the temperature is raised to 177°C, pressure remaining constant?

Solution: $V_1 = 960$ mL; $V_2 = ?$

$T_1 = (27°C + 273) = 300$ K

$T_2 = (177°C) = 273 = 450$ K

$V_1/T_1 = V_2/T_2$

$V_2 = V_1T_2/T_1$

$V_2 = (960$ mL$)(450$ K$)/(300$ K$)$

$V_2 = 1440$ mL

Example C: 400 mL of a gas at 227°C is to be reduced to a volume of 300 mL. By what temperature must the temperature be altered, keeping pressure constant?

Solution: $V_1 = 400$ mL; $V_2 = 300$ mL

$T_1 = (227°C + 273) = 500$ K

$T_2 = ?$

$V_1/T_1 = V_2/T_2$

$T_2 = V_2T_1/V_1$

$T_2 = (300$ mL$)(500$ K$)/(400$ K$)$

$T_2 = 375$ K

Practice Problems:

Practice problem A: A certain volume of a gas is under a pressure of 900 mm of Hg. When the pressure is increased by 300 mm, the gas occupies 2700 mL. If this change occurs at a constant temperature, calculate the initial volume of the gas.

Practice problem B: A certain mass of ammonia occupies 600 mL at a certain pressure. When the pressure is changed to 4 atmospheres, ammonia occupies a volume of 2.4 litres, temperature remaining constant. What was the initial pressure?

Practice problem C: A certain mass of carbon dioxide occupies a volume of 480 litres at 1 atmosphere pressure. What pressure must be applied to confine it to a cylinder of 12 litre capacity, temperature remaining constant?

Practice problem D: A sample of oxygen gas occupies a volume of 250 mL at 740 torr pressure. What volume will it occupy at 800 torr?

Practice problem E: Fluorine gas exerts a pressure of 900 torr. When the pressure is changed to 1.50 atm, its volume is 250 mL. What is the original volume? 1 atm = 760 torr.

Practice problem F: If a gas occupies 733 cm^3 at 10.0°C, at what temperature will it occupy 950 cm^3? Assume that pressure remains constant.

Practice problem G: The volume of a given mass of gas, at 150°C, is 400 mL. At what temperature will it occupy a volume of 600 mL, at the same pressure?

Practice problem H: A given mass of a gas is at a temperature of 3°C. When the gas is heated to 95°C at a constant pressure, at occupies a volume of 460 mL. What is the initial volume of the gas?

Practice problem I: A gas occupies 560 cm^3 at 285 K. To what temperature must the gas be lowered, if it is to occupy 25.0 cm^3? Assume a constant pressure.

Practice problem J: What volume will a sample of hydrogen occupy at 28.0°C if the gas occupies a volume of 2.23 dm^3 at 0°C? Assume that the pressure remains constant.