# Subspace Clustering and Its Applications

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# Subspace Segmentation Problem and Data Clustering

Problem: Let $\mathcal{M} = \bigcup_{i=1}^{l} V_i$ where $\{V_i \subset \mathcal{H}\}_{i=1}^{l}$ is a set of subspaces of a Hilbert space $\mathcal{H}$. Let $\mathbf{W} = \left\{w_j \in \mathcal{H}\right\}_{j=1}^{m}$ be a set of data points drawn from $\mathcal{M}$:

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1. determine the number of subspaces $l$,
2. determine the set of dimensions $\{d_i\}_{i=1}^{l}$,
3. find an orthonormal basis for each subspace $V_i$,
4. collect the data points belonging to the same subspace into the same cluster.

# Example 1

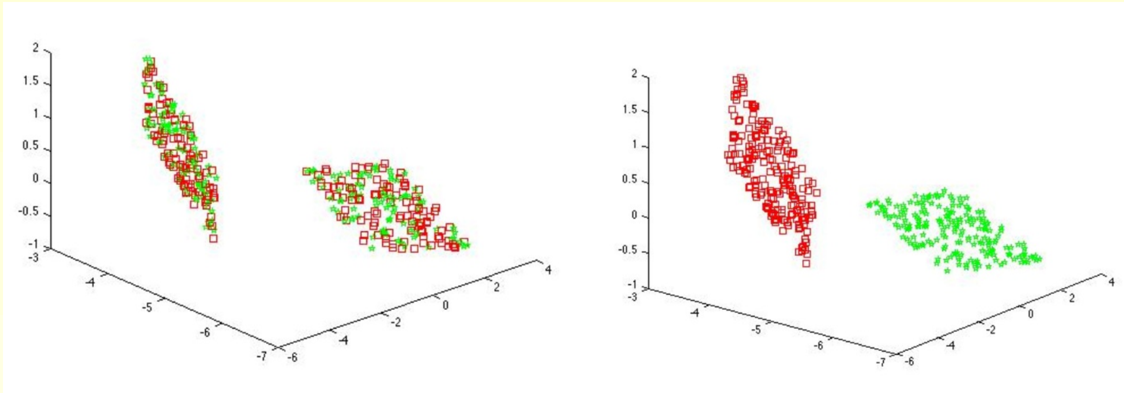

Figure 1: Data $\mathcal{F}$ belongs to two planes in $R^3$. Left panel: Starting with two initial clusters, green is supposed to belongs one cluster, while red is supposed to belong to another cluster. Right panel: Solution after clustering.

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- Learning theory: Geometry of high dimensional data, ... (Coifman, Maggioni, Lerman, ...)

# Noisy

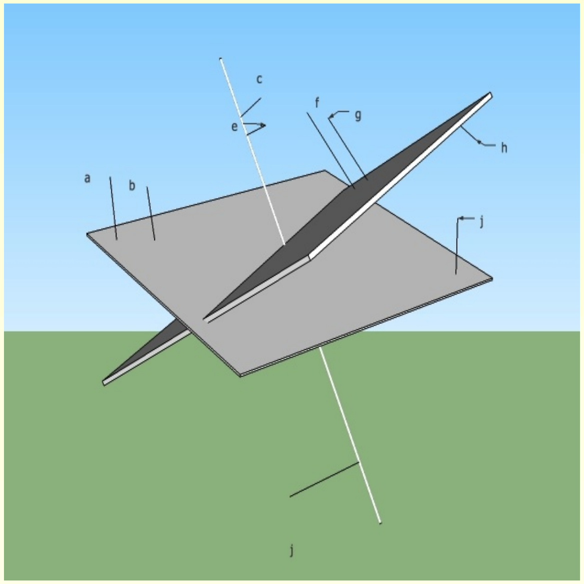


Figure 2: Non-ideal data

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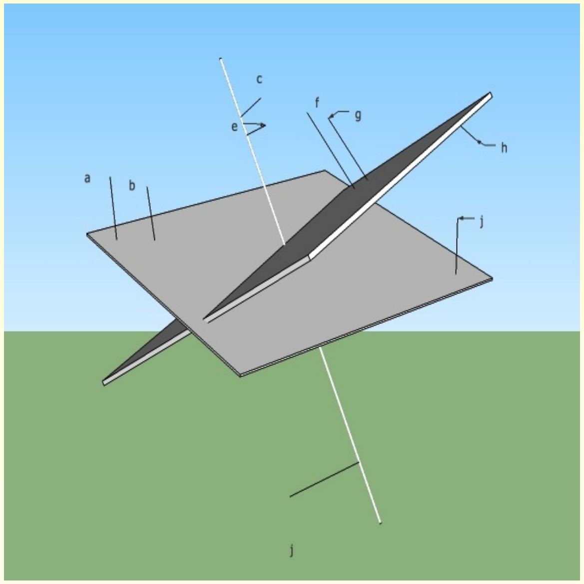


Figure 2: Non-ideal data

Data can be corrupted by noise, may have outliers or the data may not be complete, e.g., there may be missing data points.

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3. Computer Vision: Motion tracking

# Existence of solution

Given a set of vectors $\mathcal{F} = \{f_1, \ldots, f_m\}$ (the observed data) in some Hilbert space $\mathcal{H}$, find a sequence of subspaces $\{V_1, \ldots, V_l\} \subset \mathcal{C}$ minimizing

$$e(\mathcal{F}, \{V_1, \ldots, V_l\}) = \sum_{f \in \mathcal{F}} \min_{1 \le j \le l} d^2(f, V_j),$$

where the class $\mathcal{C}$ is a set of closed subspaces of $\mathcal{H}$, e.g., if $\mathcal{H} = \mathbb{R}^N$, $\mathcal{C}$ the set of subspaces of dimension less than or equal to $r$, then

$$\{V_1^0, \ldots, V_l^0\} = \text{argmin}\big\{e(\mathcal{F}, \{V_1, \ldots, V_l\}) : V_i \subset \mathbb{R}^N, \dim V_i \le r\big\}$$

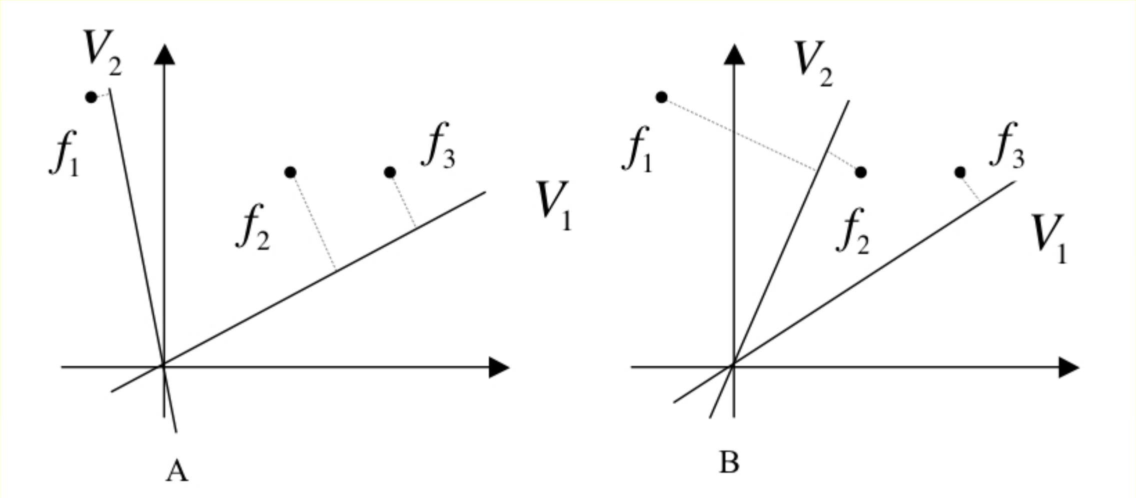


Figure 3: $\mathcal{C} = \{$subspaces of dimensions $\leq 1\}$. Objective function: A) $e = d^2(f_1, V_2) + d^2(f_2, V_1) + d^2(f_3, V_1)$; and B) $e = d^2(f_1, V_2) + d^2(f_2, V_2) + d^2(f_3, V_1)$. Configuration of $V_1, V_2$ in Panel A forced the partition $P_1 = \{f_1\}$ and $P_2 = \{f_2, f_3\}$, while the configuration in B forced the partition $P_1 = \{f_1, f_2\}$ and $P_2 = \{f_3\}$.

# Characterization in Finite d

$$e(\mathcal{F}, \{V_1, \ldots, V_l\}) = \sum_{f \in \mathcal{F}} \min_{1 \leq j \leq l} d^2(f, V_j), \quad V_i \in \mathcal{C}$$

$\mathcal{C}$ is any class of subspaces of $\mathcal{H}$. Each $V_i$ is an element of $\mathcal{C}$. View $\mathcal{C}$ as a set of projectors with the weak op. topology. $\mathcal{C}+ = \mathcal{C} +$ positive operators.

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**Theorem 1** *(A.A., Romain Tessera, FoCM 2011) Suppose $\mathcal{H}$ has dimension $d$. Then TFAE*

*(i) A solution exists in the class $\mathcal{C}$;*

*(ii) $co(\mathcal{C}^+) = co\left(\overline{\mathcal{C}^+}\right)$;*

*(iii) $\mathcal{C}^+$ is closed.*

# Signal Models

Find $\mathcal{M} = \bigcup_{i=1}^{l} V_i$ that best describe a class of images or signals from the observation of a set $\mathcal{F}$ of $m$ images.

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In most applications, the $V_i$'s describing the underlying class of signal are Shift-Invariant Spaces:

$$V(\Phi) := \text{closure}_{L_2}\, \text{span}\{\varphi_i(x-k) : \; i = 1, \ldots, n, \; k \in \mathbb{Z}^d\} \tag{1}$$

$$\mathcal{V}_n := \{\text{the set of all SIS with n generators}\}$$

# Signal models for a single subspace

Given a set of functions $\mathcal{F} = \{f_1, \ldots, f_m\} \subset L^2(\mathbb{R}^d)$, find the shift-invariant space $V$ with $n$ generators $\{\varphi_1, \ldots, \varphi_n\}$ that is "closest" to the functions of $\mathcal{F}$ in the sense that

$$V = \text{argmin}_{V' \in \mathcal{V}_n} \sum_{i=1}^{m} w_i \| f_i - P_{V'} f_i \|^2,$$

where $w_i$s are positive weights, and $\mathcal{V}_n$ is the set of all shift-invariant spaces that can be generated by $n$ or less generators.

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Although the set $\mathcal{F} = \{f_1, \ldots, f_m\}$ is finite the search is over infinite dimensional spaces in $\mathcal{V}_n$.

**Theorem 2** *(A.A., Cabrelli, Hardin, Molter, ACHA 2007)*

*Let* $\mathcal{F} = \{f_1, \ldots, f_m\}$ *be a set of functions in* $L^2(\mathbb{R}^d)$. *Then there exists* $V \in \mathcal{V}_n$ *such that*

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V'} f_i\|^2, \quad \forall\ V' \in \mathcal{V}_n$$

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3. Solution to the Union of Subspaces exists and is based parts 1 and 2.

# Video motion

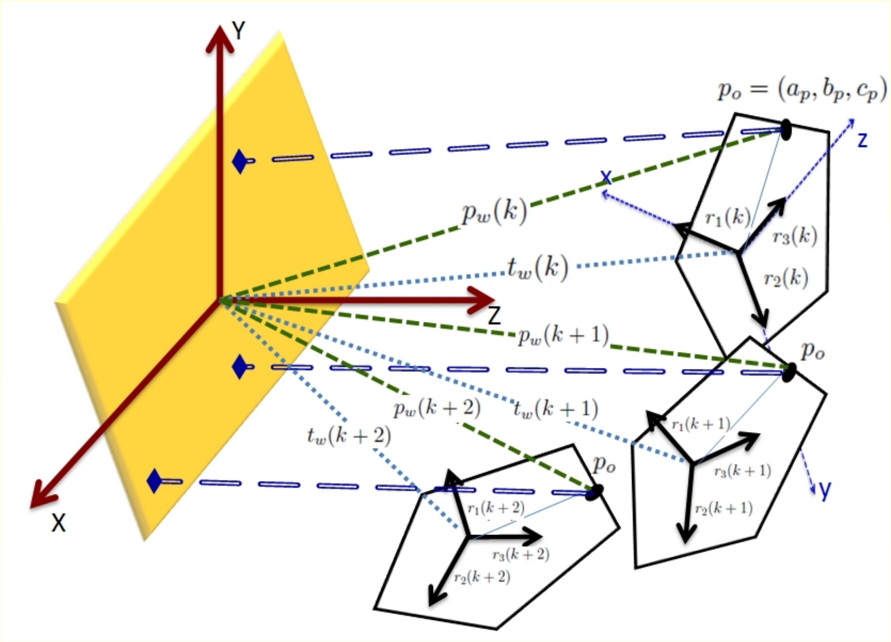


Figure 4: Affine projection of path.

Let the "trajectory" of $p$ be $w_1 = w(p) = (X_1(p), Y_1(p), \ldots, X_N(p), Y_N(p))^t$ in $\mathbb{R}^{2N}$, where $N$ is the number of frames. Then $V = \text{span}\{w_1, w_2, \ldots, w_k\}$ has $\dim V \leq 4$.

# Video motion

Let $W = [w_1, w_2, \ldots, w_m]$ be a matrix whose columns are "trajectories" of $m$ points of $l$ moving objects. Then, the columns of $W$ belong to a union of $l$ subspaces $\mathcal{M} = \bigcup_{i=1}^{l} V_i$ with $\dim V_i \le 4$.

(Loading Art2.mp4)

# Methods

- Sparsity methods (Elhamifar, Vidal, Ma, Soltanolkotabi, Candes, ...)

- Algebraic methods. e.g., Generalized Principle component Analysis (GPCA) (Vidal, Ma, Sastry,...)

- Variational methods, e.g., non-linear least squares, K-subspaces (Tseng, AA, Cabrelli, Molter, Lerman, ...)

- Statistical, e.g., Multi Stage Learning (MSL), Random Sample Consensus (RANSAC) (Fischler, Bolles,...)

- Spectral Clustering Methods, e.g., local affinity method (Yan Pollefeys, Lin, Zha, Chen Atev, Lerman, Szlam, AA, Sekmen...)

# RREF

$\mathcal{B} = \mathbb{R}^D$, $\mathcal{F} = \{w_j\}_{j=1}^N \subset \bigcup_{i \in I} S_i$, $W = [w_1 \cdots w_N] \in \mathbb{R}^{D \times N}$

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Suppose that the row reduced echelon form (Gaussian elimination) gives

$$\text{rref}(W) = \begin{pmatrix} I_1 & 0 & \dots & 0 & X & 0 & \dots & 0 \\ 0 & I_2 & \dots & \vdots & 0 & X & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & I_M & 0 & 0 & 0 & X \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

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There exists a permutation $P$ such that

$$\text{rref}(W) = [\text{Block resolved}]P,$$

if data is generic and subspaces are independent.

# data

$$W = \begin{bmatrix} 2872 & 138 & 342 & 263 & 1956 & 2016 & 1793 & 801 & 195 & 360 & 1076 & 1882 & 1918 & 2350 & 83 \\ 4041 & 249 & 467 & 516 & 129 & 288 & 2612 & 769 & 312 & 174 & 241 & 176 & 3019 & 3270 & 219 \\ 2906 & 4292 & 352 & 7240 & 2861 & 3072 & 1847 & 665 & 6968 & 646 & 1709 & 2794 & 2080 & 2366 & 1012 \\ 5803 & 1405 & 657 & 2498 & 549 & 864 & 3854 & 687 & 2158 & 390 & 629 & 628 & 4711 & 4654 & 545 \\ 5124 & 744 & 2092 & 1335 & 662 & 1056 & 2835 & 1116 & 1131 & 484 & 774 & 762 & 4867 & 4546 & 309 \\ 6701 & 3192 & 757 & 5420 & 775 & 1248 & 4502 & 578 & 5148 & 578 & 919 & 896 & 5638 & 5354 & 812 \\ 7102 & 1625 & 802 & 2862 & 888 & 1440 & 4793 & 522 & 2522 & 672 & 1064 & 1030 & 6059 & 5666 & 585 \\ 495 & 223 & 117 & 577 & 322 & 960 & 266 & 247 & 169 & 668 & 866 & 520 & 275 & 430 & 388 \\ 1184 & 2910 & 192 & 8282 & 435 & 1152 & 755 & 200 & 1482 & 762 & 1011 & 654 & 951 & 970 & 6320 \\ 2065 & 1117 & 287 & 3027 & 4040 & 4800 & 1376 & 159 & 715 & 1360 & 2920 & 4100 & 1797 & 1662 & 2172 \end{bmatrix}$$

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Binary reduced row echelon form of $W$ is $W_b$:

$$W_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Noisy data

$$\text{rref}(W) = \begin{pmatrix} I_1 & x & \dots & x & X & x & \dots & x \\ x & I_2 & \dots & \vdots & x & X & \dots & x \\ x & x & \ddots & x & x & x & \ddots & x \\ \vdots & \vdots & \ddots & I_M & x & x & x & X \\ x & x & \dots & x & x & x & x & x \end{pmatrix} \tag{3}$$

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Threshold?

$$\text{rref}(W) = \begin{pmatrix} I_1 & 0 & \dots & 0 & X & 0 & \dots & 0 \\ 0 & I_2 & \dots & \vdots & 0 & X & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & I_M & 0 & 0 & 0 & X \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

# Noisy data

$A^{-1}W$ = rref$(W)$ $\Rightarrow$ sensitive to $\sigma_{min}(A) = \|A^{-1}\|$. Estimation of $\sigma_{min}(A)$ in terms of relative positions of subspaces:

**Theorem 3** *(A, Ali Sekmen) Assume* $d_i = \dim S_i$ *and* $D = \sum_i d_i$. *Let* $\{\theta_j(S_i)\}_{j=1}^{\min(d_i, D-d_i)}$ *be the principle angles between* $S_i$ *and* $\sum_{\ell \neq i} S_\ell$. *Then*

$$\sigma_{\min}^2(A) \leq \min_i \left( \prod_{j=1}^{\min(d_i, D-d_i)} \Big(1 - \cos^2(\theta_j(S_i))\Big) \right)^{1/D}, \quad (5)$$

*where* $\sigma_{\min}(A)$ *is the smallest singular value of* $A$.

# Performance of rref

Thresholding rref algorithms does not work very well in the presence of noise. It can be used as initial stage of an iterative algorithms.

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Thresholding rref algorithms does not work very well in the presence of noise. It can be used as initial stage of an iterative algorithms.

Is there a way of modifying rref algorithm to get good segmentation even in the presence of moderate noise?

# Nearness to Local Subspace Algorithm

$W$ an $m \times N$ data matrix with columns from a union of $l$ subspaces of dimensions at most $d$, possibly noisy.

1. Dimensionality reduction: $W \rightarrow V_r$ $(r \approx l \times d << N)$
2. Normalization: Place data of $V_r$ on sphere $\mathbb{S}^{r-1}$
3. Local Subspace Estimation: For each $x_i \in \mathbb{S}^{r-1}$ find a d-dim. subspace $L_{loc}(x_i)$ nearest to $\{x_i, x_{i_1}, ..., x_{i_k}\}$ consisting of $x_i$ and $k \geq d$ closest neighbors.
4. Similarity matrix $S$: Construct an $N \times N$ similarity matrix based on $L_{loc}(x_i)$.

5. Spectral clustering: Problem is again a subspace segmentation for similarity matrix $S$, but each subspace is 1-dim. and there are $l$ subspaces. Apply SVD to obtain $S_l = U_l \Sigma_l V_l^t$ and Cluster the columns of $V_l^t$ $(l \times N)$ using k-means.

# Testing on video motion

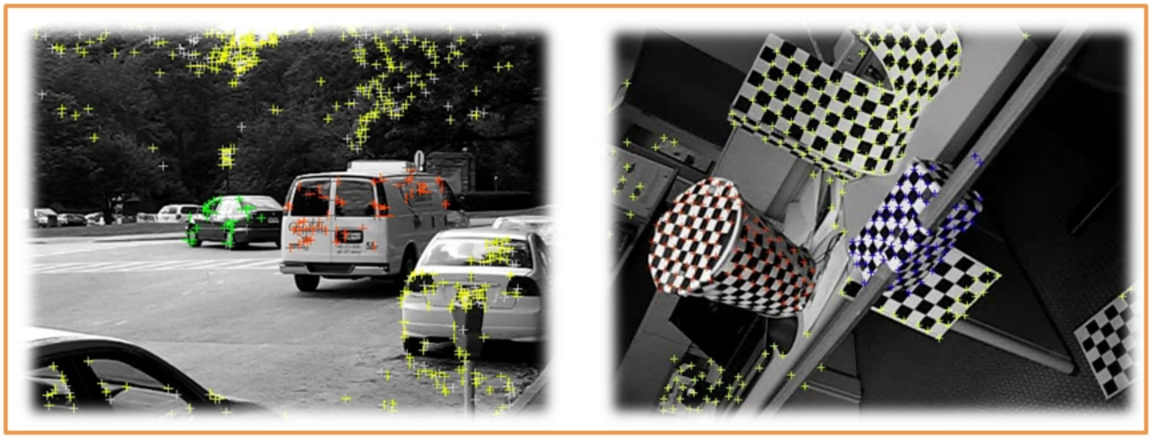

Figure 5: Samples from the Hopkins 155 Dataset.

# Testing on video motion

| Checker (78) | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
|---|---|---|---|---|---|---|---|---|
| Average | 6.09% | 2.57% | 6.52% | 4.46% | 1.55% | 0.83% | 1.12% | 0.23% |
| Median | 1.03% | 0.27% | 1.75% | 0.00% | 0.29% | 0.00% | 0.00% | 0.00% |
| **Traffic (31)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 1.41% | 5.43% | 2.55% | 2.23% | 1.59% | 0.23% | 0.02% | 1.40% |
| Median | 0.00% | 1.48% | 0.21% | 0.00% | 1.17% | 0.00% | 0.00% | 0.00% |
| **Articulated (11)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 2.88% | 4.10% | 7.25% | 7.23% | 10.70% | 1.63% | 0.62% | 1.77% |
| Median | 0.00% | 1.22% | 2.64% | 0.00% | 0.95% | 0.00% | 0.00% | 0.88% |
| **All (120 seq)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 4.59% | 3.45% | 5.56% | 4.14% | 2.40% | 0.75% | 0.82% | 0.57% |
| Median | 0.38% | 0.59% | 1.18% | 0.00% | 0.43% | 0.00% | 0.00% | 0.00% |

% classification errors for sequences with two motions.

# Three motions

| Checker (26) | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
|---|---|---|---|---|---|---|---|---|
| Average | 31.95% | 5.80% | 25.78% | 10.38% | 5.20% | 4.49% | 2.97% | 0.87% |
| Median | 32.93% | 1.77% | 26.00% | 4.61% | 0.67% | 0.54% | 0.27% | 0.35% |
| **Traffic (7)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 19.83% | 25.07% | 12.83% | 1.80% | 7.75% | 0.61% | 0.58% | 1.86% |
| Median | 19.55% | 23.79% | 11.45% | 0.00% | 0.49% | 0.00% | 0.00% | 1.53% |
| **Articulated (2)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 16.85% | 7.25% | 21.38% | 2.71% | 21.08% | 1.60% | 1.60% | 5.12% |
| Median | 16.85% | 7.25% | 21.38% | 2.71% | 21.08% | 1.60% | 1.60% | 5.12% |
| **All (35 seq)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
| Average | 28.66% | 9.73% | 22.94% | 8.23% | 6.69% | 3.55% | 2.45% | 1.31% |
| Median | 28.26% | 2.33% | 22.03% | 1.76% | 0.67% | 0.25% | 0.20% | 0.45% |

% classification errors for sequences with three motions.

# Two and Three Motions

| **All (155 seq)** | GPCA | LSA | RANSAC | MSL | ALC | SSC-B | SSC-N | NLS |
|---|---|---|---|---|---|---|---|---|
| Average | 10.34% | 4.94% | 9.76% | 5.03% | 3.56% | 1.45% | 1.24% | 0.76% |
| Median | 2.54% | 0.90% | 3.21% | 0.00% | 0.50% | 0.00% | 0.00% | 0.20% |

% classification errors for all sequences.

Review article (Open access): A Review of Subspace Segmentation: Problem, Nonlinear Approximations, and Applications to Motion Segmentation

http://www.hindawi.com/journals/isrn/2013/417492/

Thank you