Shift-Invariant Spaces with Extra Invariance

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Shift Invariant Spaces

- $L^2(\mathbb{R}^d) = \{f : \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty\}$
- T_k, defined by $T_k f(x) = f(x-k)$ is called a *Shift Operator*.
- A closed subspace $V \subset L^2(\mathbb{R}^d)$ is called a *Shift Invariant Space* if for any $f \in V$ and any $k \in \mathbb{Z}^d$, $T_k f \in V$. In other words, V is closed under shifts by integers.

Given $F \subset L^2(\mathbb{R}^d)$, we can build a SIS, $V(F)$.

- Let $\mathcal{T}(F) = \{T_k f_j : k \in \mathbb{Z}^d, f_j \in F\}$.
- Then let $V(F)$ be the closed span of $\mathcal{T}(F)$, i.e.

$$V(F) = \overline{\text{span}(\mathcal{T}(F))}^{L^2(\mathbb{R}^d)}.$$

- The elements of F are called generators of $V(F)$.
- If F is finite, $V(F)$ is called finitely generated.
- If $F = \{f\}$ then $V(F) = V(f)$ is called singly generated or principal.

- SIS are often used as approximation spaces in numerical analysis.
- For algorithms involving singly generated SIS, only one function needs to be stored.
- Many properties of $V(F)$ can be traced back to properties of F.

- Given a lattice $\mathbb{Z}^d \subset \Gamma \subset \mathbb{R}^d$, $V(F)$ is said to be Γ-*invariant* if $f \in V(F)$ implies that $T_\gamma f \in V(F)$ for all $\gamma \in \Gamma$.
- $V(F)$ is said to be *translation invariant* if $f \in V(F)$ implies that $T_t f \in V(F)$ for all $t \in \mathbb{R}^d$, i.e if $V(F)$ is $\mathbb{R}^d$-invariant.

Riesz Bases and Frames

- A collection $\{h_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a *Riesz Basis* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall c = (c_n) \in l^2(\mathbb{N}), \quad A\|c\|^2_{l^2(\mathbb{N})} \leq \|\sum_{n=1}^{\infty} c_n h_n\|^2 \leq B\|c\|^2_{l^2(\mathbb{N})}.$$

- A collection $\{h_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall h \in \mathcal{H}, \quad A\|h\|^2_{\mathcal{H}} \leq \sum_{n=1}^{\infty} |\langle h, h_n\rangle|^2 \leq B\|h\|^2_{\mathcal{H}}.$$

Examples of SIS

- Paley Wiener Space - $\{g \in L^2(\mathbb{R}^d) : \widehat{g}(\xi) = 0 \text{ for } |\xi| > \frac{1}{2}\}$.
- Splines with integer knots contained in $L^2(\mathbb{R})$ - S_n^{n-1}
- Gabor Spaces - $\overline{\text{span}\{f(x-k)e^{2\pi inx} : n,k \in \mathbb{Z}\}}$.
- Wavelet Spaces - $\overline{\text{span}\{a^{-k/2}f(a^{-k}x - nb) : n,k \in \mathbb{Z}\}}$.

Examples of Singly Generated SIS

- Paley Wiener Space - $\{g \in L^2(\mathbb{R}^d) : \widehat{g}(\xi) = 0 \text{ for } |\xi| > \frac{1}{2}\}$.
- Splines with integer knots contained in $L^2(\mathbb{R})$
- Gabor Spaces - $\overline{\text{span}\{f(x-k)e^{2\pi i n x} : n, k \in \mathbb{Z}\}}$.
- Wavelet Spaces - $\overline{\text{span}\{a^{-k/2} f(a^{-k}x - nb) : n, k \in \mathbb{Z}\}}$.

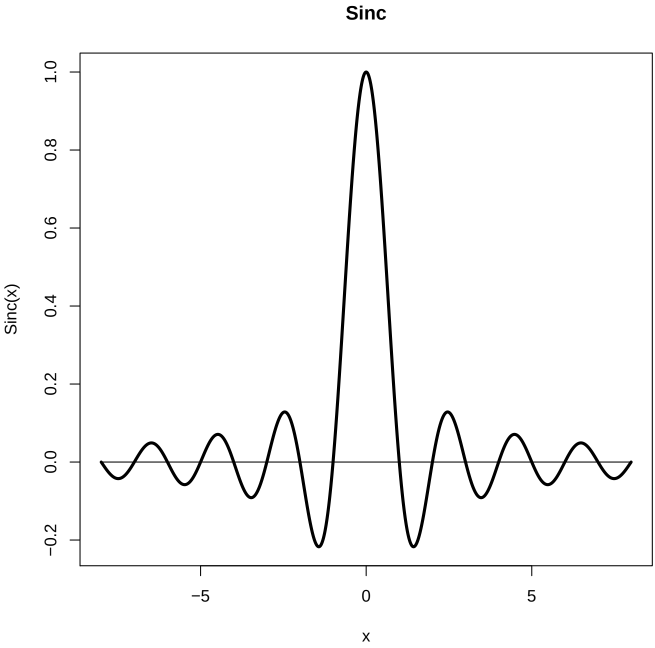
Paley Wiener Space, PW, or the Space of Bandlimited Functions

- $g \in PW \iff$ the Fourier Transform of g given by,

$$\widehat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{2\pi i \xi x} dx$$

  is zero outside of $[-\frac{1}{2}, \frac{1}{2}]$.
- Let $f = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.
- Then PW is generated by f, or

$$PW = V(f).$$


Sinc
Sinc(x)
1.0
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0.4
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5
x

- If $g \in PW$, and $\lambda \in \mathbb{R}$, then $T_\lambda g(x) = g(x-\lambda)$ is also in PW.
- This is true since

$$\widehat{T_\lambda g}(\xi) = e^{2\pi i \lambda \xi} \widehat{g}(\xi).$$

- Thus, PW is translation invariant.

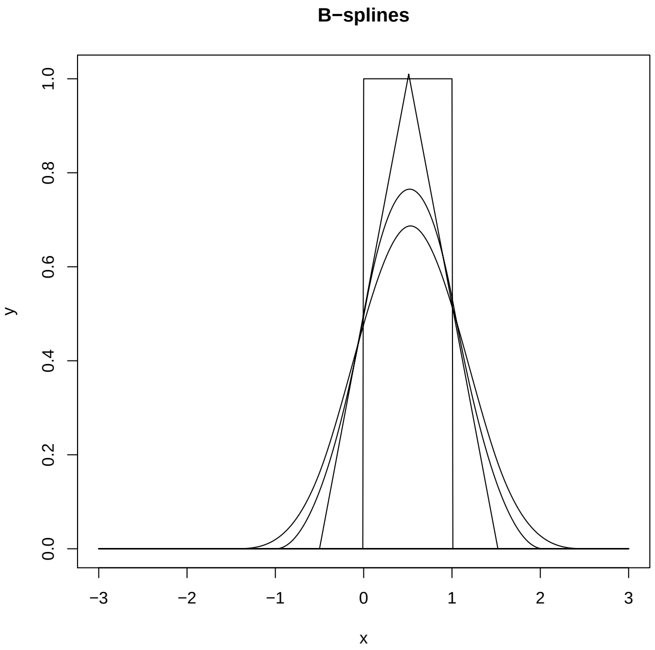
Spline Spaces - S_n^{n-1}

- $f \in S_n^{n-1} \iff f$ restricted to $[k, k+1]$ is a polynomial of degree n, $f \in L^2(\mathbb{R})$, and $f \in C^{n-1}(\mathbb{R})$.
- Let β_0 be the characteristic function of $[0,1]$, and iteratively define

$$\beta_n(x) = \int_{\mathbb{R}} \beta_{n-1}(x-t)\beta_0(t)dt.$$

- Then S_n^{n-1} is a SIS generated by β_n, or

$$S_n^{n-1} = V(\beta_n).$$


B−splines
y
x
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1
2
3

- For some $f \in S_n^{n-1}$ consider shifting f by a non-integer $\lambda \in \mathbb{R}$.
- This has the effect of shifting the knots into the interior of the intervals.
- Clearly, we can find some $f \in S_n^{n-1}$ where $T_\lambda f \notin S_n^{n-1}$.
- Thus, S_n^{n-1} has no extra invariance.

Key Points

- PW is translation invariant, and the generator decays slowly (like $\frac{1}{x}$).
- S_n^{n-1} has compactly supported generators, and it has no extra invariance.
- It turns out that this kind of behavior holds in general and not just for these two examples.

Heisenberg Uncertainty Principle

- An uncertainty principle in harmonic analysis is a result which constrains how well-localized a function f or its Fourier transform $\widehat{f}$ can be.
- A famous uncertainty principle is given by the d-dimensional Heisenberg inequality: $\forall f \in L^2(\mathbb{R}^d)$,

$$\left(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}^d} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi\right) \geq \frac{d^2}{16\pi^2} \|f\|^4_{L^2(\mathbb{R}^d)}.$$

Balian-Low Theorem

- The Balian-Low Theorem for Gabor Systems is another example of an Uncertainty Principle.
- Given $f \in L^2(\mathbb{R})$ the associated *Gabor system* $\mathcal{G}(f, 1, 1) = \{f_{m,n}\}_{m,n\in\mathbb{Z}}$ is defined by $f_{m,n}(x) = e^{2\pi i m x} f(x - n)$.

Theorem (Balian-Low Theorem)

Let $f \in L^2(\mathbb{R})$. If $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi\right) = \infty.$$

BLT for SIS

- In fact, Balian-Low type theorems exist for generators of Shift Invariant Spaces.

Theorem (Aldroubi, Sun, Wang-2010)

Suppose that $f \in L^2(\mathbb{R})$ and that $\mathcal{T}(f)$ is a Riesz Basis for $V(f)$. If $V(F)$ is $\frac{1}{n}\mathbb{Z}$-invariant then for all $\epsilon > 0$,

$$\int_{\mathbb{R}} |x|^{1+\epsilon} |f(x)|^2 dx = \infty.$$

Theorem (Tessera, Wang-2013)

Let $f \in L^2(\mathbb{R}^d)$ and $\mathbb{Z}^d \subset \Gamma$ is a lattice. Suppose that $\mathcal{T}(f)$ is a frame for $V(f)$. If $V(f)$ is Γ-invariant then for any $\epsilon > 0$,

$$\int_{\mathbb{R}^d} |x|^{d+\epsilon} |f(x)|^2 dx = \infty.$$

Main Result

Theorem (Hardin, Northington, Powell)

Fix a lattice $\mathbb{Z}^d \subset \Gamma \subset \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$. Suppose that $\mathcal{T}(f)$ is a frame for $V(f)$. If $V(f)$ is Γ-invariant then

$$\int_{\mathbb{R}^d} |x| \, |f(x)|^2 dx = \infty.$$

Smoothness vs Decay

Consider the following for $f \in L^2(\mathbb{R})$.

$$\widehat{f'}(\xi) = \int_{\mathbb{R}^d} f'(x) e^{2\pi i x \xi} dx$$

$$= (2\pi i \xi) \int_{\mathbb{R}^d} f(x) e^{2\pi i x \xi} dx$$

$$(2\pi i \xi) \widehat{f}(\xi).$$

Fractional Sobolev Spaces

- For $s > 0$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of all measurable functions f defined on $\mathbb{R}^d$ such that $f \in L^2(\mathbb{R}^d)$ and

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty. \qquad (1)$$

Main Result

Theorem (Hardin, Northington, Powell)

Fix a lattice $\mathbb{Z}^d \subset \Gamma \subset \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$. Suppose that $\mathcal{T}(f)$ is a frame for $V(f)$. If $V(f)$ is Γ-invariant then

$$\int_{\mathbb{R}^d} |x| \, |f(x)|^2 dx = \infty.$$

In other words, the generator satisfies $\widehat{f} \notin H^{1/2}(\mathbb{R}^d)$.

- Given a lattice $\Gamma \subset \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, the *Periodization* of f is defined as

$$P_\Gamma(f)(x) = \sum_{\gamma \in \Gamma} |f(x+\gamma)|^2.$$

- The following calculation shows that the inner products of elements of $\mathcal{T}(f)$ are encoded in $P_{\mathbb{Z}^d}(\widehat{f})$

$$
\begin{aligned}
\langle T_k f, T_{k'} f\rangle_{L^2(\mathbb{R}^d)} &= \langle e^{-2\pi i \xi\cdot k}\widehat{f}, e^{-2\pi i\xi\cdot k'}\widehat{f}\rangle_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot(k-k')}\widehat{f}(\xi)\overline{\widehat{f}(\xi)}d\xi \\
&= \sum_{l\in\mathbb{Z}^d}\int_{[0,1]^d} e^{-2\pi i\xi\cdot(k-k')}\widehat{f}(\xi-l)\overline{\widehat{f}(\xi-l)}d\xi \\
&= \int_{[0,1]^d} e^{-2\pi i\xi\cdot(k-k')}P_{\mathbb{Z}^d}(\widehat{f})(\xi)d\xi \\
&= \widehat{P_{\mathbb{Z}^d}(\widehat{f})}(k-k')
\end{aligned}
$$

Proposition (de Boor, DeVore, Ron)

Given $f \in L^2(\mathbb{R}^d)$, $\mathcal{T}(f) = \{T_k f : k \in \mathbb{Z}^d\}$ forms a Riesz basis for $V(f)$ if and only if there exists $s \geq 1$ such that

$$s^{-1} \leq P_{\mathbb{Z}^d}(\widehat{f})(\xi) \leq s \quad \textit{a.e. } x \in [0,1]^d. \tag{2}$$

Proposition (Bownik)

$\mathcal{T}(f)$ forms a frame for $V(f)$ if and only if there exists $s \geq 1$ such that for almost every $x \in [0,1]^d$,

$$s^{-1} P_{\mathbb{Z}^d}(\widehat{f})(\xi) \leq (P_{\mathbb{Z}^d}(\widehat{f})(\xi))^2 \leq s P_{\mathbb{Z}^d}(\widehat{f})(\xi). \tag{3}$$

Theorem (Aldroubi, Cabrelli, Heil, Kornelson, Molter)

Let $\mathbb{Z}^d < \Gamma$ be lattices in $\mathbb{R}^d$. Let $R \subset \mathbb{Z}^d$ be a collection of representatives of the quotient $\mathbb{Z}^d/\Gamma^*$ so that

$$P_{\mathbb{Z}^d}(\widehat{f})(\xi) = \sum_{k \in R} P_{\Gamma^*}(\widehat{f})(x+k), \; a.e. \; x \in \mathbb{R}^d.$$

The space $V(f)$ is Γ-invariant if and only if for almost every $x \in \mathbb{R}^d$, at most one of the terms in the right hand sum is nonzero.

Fourier coefficients

The Fourier coefficients of $f \in L^2([0,1]^d)$ are defined $\forall \xi \in \mathbb{Z}^d$,

$$\widehat{f}(\xi) = \int_{[0,1]^d} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$

Also recall Parseval's theorem

$$\int_{[0,1]^d} |f(x)|^2 dx = \sum_{\xi \in \mathbb{Z}^d} |\widehat{f}(\xi)|^2 \tag{4}$$

Periodic Sobolev Space

Given $s > 0$, define the Sobolev space
$H^s([0,1]^d) = \{f \in L^2([0,1]^d) : \|f\|_{\dot{H}^s([0,1]^d)} < \infty\}$, where

$$\|f\|^2_{\dot{H}^s([0,1]^d)} = \sum_{\xi \in \mathbb{Z}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2.$$

Theorem (Hardin, Northington, Powell)

Let $0 < s < 1$. If $f \in H^s(\mathbb{R}^d)$ and $P_{\mathbb{Z}^d}(f)$ is bounded, then

$$P_{\mathbb{Z}^d}(f) \in H^s([0,1]^d).$$

Lemma ($\mathbb{R}^d$ Proof by Bourgain, Brezis, Nirenberg)

If $g \in H^{1/2}([0,1]^d)$ and for almost every $x \in [0,1]^d$, either $g(x) = 0$, or $g(x) \geq C > 0$ then either $g(x) = 0$ a.e. or $g(x) \geq C$ a.e.

- Assume $\widehat{f} \in H^{1/2}(\mathbb{R}^d)$.
- Then $P_{\mathbb{Z}^d}(\widehat{f}), P_{\Gamma^*}(\widehat{f}) \in H^{1/2}([0,1]^d)$.
- Since $\mathcal{T}(f)$ forms a frame, $P_{\mathbb{Z}^d}(\widehat{f})$ is either zero, or bounded away from zero.
- Since $P_{\mathbb{Z}^d}(\widehat{f})(\xi) = \sum_{k \in R} P_{\Gamma^*}(\widehat{f})(x+k)$, we must have that $P_{\Gamma^*}(\widehat{f})$ is zero on a large set, and bounded away from zero on a large set.
- This contradicts the previous lemma.